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Matrix strings in a B-field

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Abstract: We study the discrete light-cone quantization (DLCQ) of closed strings in the background of Minkowski space-time and a constant Neveu-Schwarz B-field. For the Bosonic string, we identify the B-dependent part of the thermodynamic free energy to all orders in string perturbation theory. For every genus, B appears in a constraint in the path integral which restricts the world-sheet geometries to those which are branched covers of a certain torus. This is the extension of a previous result where the B-field was absent [1]. We then discuss the coupling of a B-field to the Matrix model of M-theory. We show that, when we consider this theory at finite temperature and in a finite B-field, the Matrix variables are functions which live on a torus with the same Teichmüller parameter as the one that we identified in string theory. We show explicitly that the thermodynamic partition function of the Matrix string model in the limit of free strings reproduces the genus 1 thermodynamic partition function of type-IIA string. This is strong evidence that the Matrix model can reproduce perturbative string theory. We also find an interesting behavior of the Hagedorn temperature.

Keywords: Bosonic Strings, Superstrings and Heterotic Strings, M(atrix) Theories.

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1. Introduction and summary

It is widely believed that the five different perturbatively consistent superstring theories describe special points in the space of vacua of a single dynamical structure called M-theory. However, the nature of the degrees of freedom and a detailed description of the dynamics of M-theory are as yet unknown. What has recently become apparent is that there are other points, besides the perturbative string limits, in the moduli space of vacua where M-theory could be understood. One of them is the infinite momentum frame which is conjectured to be described by the Matrix model [2, 3]. Others are the various strong field limits which produce field theories with non-commutative geometry [4], or non-commutative open string (NCOS) theory [5, 6] or wrapped and non-relativistic closed string theory models [7, 8, 9]. In this paper we will compare two limits of M-theory in an overlapping domain of validity — the Matrix model in the limit where it should produce perturbative string theory and perturbative string theory in the kinematical context, discrete light-cone quantization (DLCQ), which is described by the Matrix model. We will consider each model in a background of a constant Neveu-Schwarz antisymmetric tensor field, $B_{\mu\nu}$. The coupling of this field to each model is simple and its effect can be taken into account exactly. This will give a one-parameter comparison of the two theories and we will find remarkable agreement.

The Matrix model [2, 3] (for a recent review see [10]) with $N \times N$ matrices and gauge group SU(N) is conjectured to describe M-theory on a background of eleven-dimensional Minkowski space with a compact null direction, $X^+ \sim X^+ + 2\pi R$ and N units of null momentum, $P^- = N/R$.

Type-IIA superstrings can be obtained from M-theory by compactifying a space direction. This compactification adds a dimension to the matrix model [11]. The resulting Matrix string theory [12, 13] is $1+1$-dimensional maximally supersymmetric Yang-Mills theory. With the appropriate identification of degrees of freedom, it should be a non-perturbative formulation of type-IIA superstrings on a space with a compact null direction. Dijkgraaf, Verlinde and Verlinde [13] argued that perturbative string theory is described by the moduli space of classical vacua of the $1+1$-dimensional Yang-Mills theory. In this moduli space, the degrees of freedom are mutually commuting matrices. The string degrees of freedom are the simultaneous eigenvalues of the matrices. The correct description of the dynamics of these eigenvalues in the limit which produces free strings is a super-conformal field theory on a symmetric orbifold $(\mathbb{R}^8)^N/S_N$. They also showed that the first correction to the free string hamiltonian is an irrelevant operator which precisely reproduces the Mandelstam three-string vertex. Elaborations on this limit have been discussed in detail in the literature [1] and [14]–[25].

In the following, we will provide further support for these ideas by comparing that limit of matrix string theory which should produce weakly coupled strings with discrete light-cone quantized (DLCQ) perturbative string theory. We will be particularly interested in examining the effect of coupling a constant background $B_{\mu\nu}$ to the Matrix string model and comparing the perturbative string limit to the DLCQ type-IIA superstring in a background B-field. We will find a remarkable agreement between the two. This is an elaboration of our previous results in [1, 21] where a similar comparison was made in the absence of $B_{\mu\nu}$. It provides further evidence in support of the Matrix model conjecture.

We will be interested in the thermodynamic partition function of matrix string theory. To form the thermodynamic partition function, we must identify the energy. In the Matrix model, one of the light-cone momenta is given by $P^- = N/R$ where R is the compactification radius of the light-cone and the other one is given by $P^+ = H$, the hamiltonian of $1+1$-dimensional supersymmetric Yang-Mills theory with gauge group $\mathrm{U}(N)$. The energy is given by $P^0 = (P^+ + P^-)/\sqrt{2} = (N/R + H)/\sqrt{2}$ which can be used to form the partition function

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta P^0}\right) = \sum_{N=0}^{\infty} e^{-\beta N/\sqrt{2}R}\,\mathrm{Tr}\left(e^{-\beta H/\sqrt{2}}\right), \tag{1.1}$$

where $\beta = 1/T$ is the inverse temperature. The sum over N is the trace over the spectrum of P^-. The remaining trace is the thermodynamic partition function of the $1+1$-dimensional super Yang-Mills theory with gauge group $\mathrm{U}(N)$ and inverse temperature $\beta/\sqrt{2}$. It is given by the usual finite temperature field theory path integral with compact euclidean time where the supersymmetry of the model is broken by the anti-periodic boundary conditions for fermions. It will turn out that the matrix string model partition function (1.1) contains a functional integral over matrix-valued fields which live on a particular 2-torus, characterized by Teichmüller parameter

$$\tau_0 = i\frac{\left(1 - 2\pi\alpha' i B^E\right)}{\nu}\,. \tag{1.2}$$

where

$$\nu = \frac{4\pi\alpha'}{\sqrt{2}\beta R} \tag{1.3}$$

and B^E is the euclidean version of a component of the B-field. We will find that this underlying torus also emerges in the string theory in an interesting way.

We shall begin by studying the closed bosonic string and the IIA superstring in a constant external $B_{\mu\nu}$. In order to compare with the Matrix model, we do discrete light-cone quantization (DLCQ) of the string. We shall consider the thermodynamic partition function. Studying either string theory or Matrix theory at finite temperature compactifies euclidean time, $X^0 \sim X^0 + \beta$. In the IIA string theory, this gives a second compactification of the target space: both the null Minkowski direction and euclidean time are then compact. In [1, 21] it was seen that this double compactification results in an interesting constraint on geometries of world-sheets in the path integral formulation of the string. Here, we shall see how this constraint is modified by the presence of a B-field.The result will be that the integral over all geometries of Riemann surfaces in the string path integral is reduced to an integral over those Riemann surfaces which are branched covers of a torus with Teichmüller parameter given by τ_0 in (1.2).

Without compactifications of space-time, and in the absence of D-branes, closed strings would not couple to a constant B-field. In fact, a constant B-field is gauge equivalent to a constant electromagnetic field and the electric charges that would couple to it live at the ends of open strings. Closed strings are neutral and are normally unaffected by an electromagnetic field. However, when some space-time directions are compact, the states where the closed string wraps the compact dimension can couple to B. The coupling amounts to a constant shift of energies and momenta of the wrapped strings. Its origin can intuitively be understood by imagining a closed string as being made from an open string by fusing its ends. If the open string carries charges of opposite sign on its endpoints, the process of creating a small open string, wrapping it around the compact dimension, then fusing the ends together obtains a contribution to its energy from transporting the charged endpoints in a constant electric field. The energy shift is $2\pi Rn \cdot B$ where B is the component of electric field in the compact direction, R is the compactification radius and n is the number of times the resulting closed string wraps the compact direction. A constant shift in energy is a chemical potential and an electric B-field should therefore induce a finite density of wrapped closed strings. There would also be a similar shifts in momenta of wrapped states coming from the magnetic components of the B-field. If more directions were compact, the presence of the B-field would lead to a higher-dimensional non-commutative Yang-Mills theory [26, 27, 28].

Consider a closed bosonic string propagating on 26-dimensional Minkowski space with a constant background B-field. The action is given by

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma \partial^a X^\mu \partial_a X_\mu + \frac{1}{2}B_{\mu\nu}\int d^2\sigma \epsilon^{ab}\partial_a X^\mu \partial_b X^\nu \,.$$

The B-field contributes a total derivative term to the action. The equations of motion do not depend on B and have their usual form,

$$\begin{aligned} \partial_+\partial_- X^\mu &= 0\,, \\ \partial_+ X^\mu \partial_+ X_\mu &= 0\,, \qquad \partial_- X^\mu \partial_- X_\mu = 0\,. \end{aligned}$$

Noether currents can depend on B. For example, total momentum which is the Noether charge for space-time translation invariance, has the form

$$P^\mu = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma \partial_\tau X^\mu + B^\mu{}_\nu \int_0^\pi d\sigma \partial_\sigma X^\nu \tag{1.4}$$

and we see that it is modified by B only if for some μ, $\int d\sigma \partial_\sigma X^\mu \neq 0$. This can happen when the string wraps a compactified dimension.

Let us consider the simple case where one of the spatial dimensions is compact, $X^1 \sim X^1 + 2\pi R_1$ and where the only non-zero component of B is $B_{01} \equiv B$. The mass-shell and level matching conditions are

$$\left(P^0 - 2\pi B R_1 n\right)^2 - \vec{p}^{\,2} = \left(\frac{m}{R_1}\right)^2 + \left(\frac{nR_1}{\alpha'}\right)^2 + \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right), \tag{1.5}$$

$$N - \tilde{N} = mn\,. \tag{1.6}$$

Here m/R_1 is the quantized momentum in the compact dimension. The integer n is the number of times the string wraps the compact dimension. The shift of the energy by the B-dependent term is a result of the shift of the momentum in (1.4). We see that, for fixed n, the B-field affects the spectrum like a chemical potential, $2\pi B R_1 n$ in (1.5).

The mass-shell condition (1.5) is solved by light-cone momenta, $P^{\pm} = \frac{1}{\sqrt{2}}(P^0 \pm P^1)$,

$$P^- = \frac{1}{\sqrt{2}}\left(2\pi B n R_1 + \sqrt{\left(\frac{m}{R_1}\right)^2 + \left(\frac{nR_1}{\alpha'}\right)^2 + \vec{p}^{\,2} + \frac{2}{\alpha'}(N + \tilde{N} - 2)} + \frac{m}{R_1}\right),$$

$$P^+ = \frac{1}{\sqrt{2}}\left(2\pi B n R_1 + \sqrt{\left(\frac{m}{R_1}\right)^2 + \left(\frac{nR_1}{\alpha'}\right)^2 + \vec{p}^{\,2} + \frac{2}{\alpha'}(N + \tilde{N} - 2)} - \frac{m}{R_1}\right). \tag{1.7}$$

We can obtain a compactified null direction from a compactified spatial direction by an infinite boost along the compact direction together with an infinite rescaling of R_1 [29]. We consider the state where the 1-component of the momentum is large and negative and boost in the positive 1 direction. The light-cone momenta in the boosted frame are scaled by the factors

$$\tilde{P}^- = \lim_{R_1 \longrightarrow 0} \frac{R_1}{\sqrt{2}R} P^-\,, \qquad \tilde{P}^+ = \lim_{R_1 \longrightarrow 0} \frac{\sqrt{2}R}{R_1} P^+$$

to get

$$\tilde{P}^- = \frac{m}{R}\,, \qquad \tilde{P}^+ = 2\pi B n R + \frac{R}{2m}\left(\vec{p}^{\,2} + \frac{4}{\alpha'}(N + \tilde{N} - 2)\right). \tag{1.8}$$

Now, R is the light-cone compactification radius, $X^+ \sim X^+ + 2\pi R$. The B-field contributes a chemical potential-like shift to the light-cone energy P^+. The only other place that the wrapping number, n, appears is in the level matching condition (1.6) which is unchanged. Note that if we combine (1.8) and (1.6), the mass operator has the form

$$M^2 = 2P^+P^- - \vec{P}^{\,2} = \frac{2}{\alpha'}\left(1 + 2\pi\alpha' B\right)(N - 1) + \frac{2}{\alpha'}\left(1 - 2\pi\alpha' B\right)\left(\tilde{N} - 1\right). \tag{1.9}$$

In this formula, the string tensions of left and right-movers are shifted by factors of $(1 + 2\pi\alpha' B)$ and $(1 - 2\pi\alpha' B)$, respectively.

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We are interested in seeing how this spectrum arises in the appropriate limit of Matrix string theory. We shall consider the case where $B_{01} = B_{-+} \equiv B$ is the only non-zero component of the B-field and where it is a constant. To find the action of Matrix string theory in this B-field, our starting point is the action for D1-branes on a space-time with the X^1-direction compactified and with a background metric

$$G_{\mu\nu} = \begin{pmatrix} -1 + (2\pi\alpha' B)^2 & 2\pi\alpha' B & 0 & \dots \\ 2\pi\alpha' B & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.10}$$

and no B-field

$$B_{\mu\nu} = 0\,.$$

The correct D1-brane action can be found by dimensionally reducing ten-dimensional supersymmetric Yang-Mills theory with the above space-time metric to obtain maximally supersymmetric $1+1$-dimensional Yang-Mills theory whose two-dimensional space-time metric is the upper left-hand corner of $G_{\mu\nu}$.

Using the Buscher rules [30, 31] for T-duality, this is equivalent to D0-branes on a space with the same compactified direction with dual radius, the Minkowski metric $G_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1, \dots)$ and non-zero $B_{01} = B$-field.

Then, a combination of arguments following Seiberg [29] and Sen [32] and Dijkgraaf, Verlinde and Verlinde [13] can be used to show that, with the appropriate identification of parameters, this D0-brane action is also the Matrix string model which should describe Matrix strings in a B-field. Note that, under the boost to the infinite momentum frame which is required to make this identification, the component of the B-field that we are interested in, $B_{01} = B_{-+}$, is invariant.

The linearized coupling of an external B-field to both the Matrix model and the Matrix string model has been found before [33, 34]. We find that when $B_{\mu\nu}$ is a constant, the full B-dependence of the Matrix string model action has linear and quadratic terms in B. The linear term agrees with the one found in [33]. From the Matrix string action, we can also deduce how the B-field appears in the Matrix model action. There is also a linear and quadratic term and the linear term agrees with the coupling found in [34].

Weakly coupled string theory has a density of states which grows exponentially with energy [35]. This gives rise to a maximum temperature, called the Hagedorn temperature, beyond which a gas of strings cannot be heated. This Hagedorn temperature is also sometimes interpreted as a phase transition [36, 37, 38].

It is interesting to ask the question where a Hagedorn-like behavior could arise in (1.1). Since the last factor is just the partition sum of a field theory, it is clear that the only possible source of divergence which could give a Hagedorn temperature is in the summation over N. The convergence of the sum over N is governed by the large-N limit of the gauge theory. Indeed, there is a rough classification of the behaviors that can occur according to the three possible asymptotic behaviors for this sum, depending on the state of the gauge theory in the large-N limit. The free energy, F, of the gauge theory is defined by

$$\mathrm{Tr}\, e^{-\beta H/\sqrt{2}} = e^{-\beta F/\sqrt{2}}\,.$$

As a function of N in the large-N limit, the free energy can generally be expected to be negative and proportional to a power of N,

$$\beta F \sim -N^a \cdot \ldots\,.$$

There is a simple classification of the behavior of the sum over N which depends on the exponent, a,

$$\begin{aligned} &a < 1\,, && \text{confinement}\,, \\ &a = 1\,, && \beta F \sim -\frac{N\beta_H}{R}\,, \qquad \text{free strings}\,, \\ &a > 1\,, && \text{e.g. } \beta F \sim -N^2 \cdot \ldots\,, \qquad \text{deconfined}\,. \end{aligned} \tag{1.11}$$

An example of the first case is the gauge theory in a confining phase. The large-N limit of its free energy is of order one. Then, the sum over N in (1.1) will be convergent and the partition function will be field theory-like with a field theoretical asymptotic density of states.

In the second case in (1.11), which is the one that we will study in this paper, the eigenvalues of the matrices dominate and there are N of them, thus the free energy of order N. This is the limit of free long strings found by Dijkgraaf, Verlinde and Verlinde [13]. We emphasize that, in this paper, we will not discuss how this limit is obtained or whether it exists as a bona-fide phase of the $1+1$-dimensional gauge theory. We will assume that it is obtained in the limit of strong gauge theory coupling and examine the consequences. In this case, the partition sum will diverge when $\beta < \beta_H$ with $\beta_H = 1/T_H$ defining the Hagedorn temperature.

In the third case in (1.11), $\beta F \sim -N^2 \ldots$, which is what is generically expected in this supersymmetric gauge theory in the large-N limit. If the theory is in a deconfined or screening phase, the free energy is proportional to $-N^2$ in large-N 't Hooft limit where $g^2_{\rm YM}N$ is held fixed. Then, the partition sum does not converge for any value of the parameters. We regard this as a reflection of the Jeans instability of hot flat space. States of the Matrix model which have free energy of order $-N^2$ in the large-N limit correspond to string theory black holes. Divergence of the partition sum is due to the nucleation of black holes which dominate the entropy at any temperature [1]. An explicit evaluation of the perturbative limit of the Yang-Mills theory can be found in refs. [39, 40].

[1]The authors thank Moshe Rozali for an informative discussion on this point

1.1 Results

First of all, to illustrate the point, we quantize the Bosonic closed string using DLCQ and operator methods. We find the spectrum and use it to compute the thermodynamic partition function for free strings. This generalizes previous work [21] to a constant B-field. We find that, as in [21], the free energy is given by a discrete sum over Teichmüller parameters,

$$\frac{F}{V} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{s=0}^{k-1}\frac{1}{k^2}\left(\frac{1}{4\pi^2\alpha'\hat{\tau}_2}\right)^{13}\exp\left(-\frac{n^2\beta^2}{4\pi\alpha'\hat{\tau}_2}\right)\left(\eta\left(\tau_-\right)\right)^{-24}\left(\bar{\eta}\left(\tau_+\right)\right)^{-24}, \quad (1.12)$$

where

$$\begin{aligned} \tau_+ &= \frac{s}{k} + i\frac{(1+2\pi\alpha' B)}{\nu}\frac{n}{k}, \\ \tau_- &= \frac{s}{k} + i\frac{(1-2\pi\alpha' B)}{\nu}\frac{n}{k} \end{aligned} \quad (1.13)$$

and $\nu = \frac{4\pi\alpha'}{\sqrt{2}\beta R}$, the parameter in (1.3).

The integers take values $k, n = 1, 2, 3, \ldots$ and $s = 0, 1, \ldots, k-1$. Also,

$$\hat{\tau}_2 = \frac{n}{k\nu} \quad (1.14)$$

is the average of the imaginary parts of τ_+ and τ_-. In spite of the B-dependent difference between τ_+ and τ_-, this partition function is real and exhibits the expected modular invariance. It has a symmetry under the replacement

$$\sqrt{2}\pi\beta RB \longrightarrow \sqrt{2}\pi\beta RB + 2\pi i \cdot \text{ integer}.$$

This symmetry is related to a large gauge invariance of the string theory on a toroidal space, $B \to B + 2\pi i \cdot (\text{integer})/V_T$, where V_T is the volume of the torus, is a gauge transformation. The free energy exhibits a symmetry under this gauge transformation.

We then show how the same result can be obtained in a covariant quantization using the path integral where the worldsheets have genus 1. In this case the simultaneous compactification of the Minkowski space null direction and euclidean time involves an periodic identification of coordinates that contains complex numbers. It further requires that we use a euclidean B^E-field which is related to the physical Minkowski one by $B^E_{01} = -iB_{01}$. We can re-obtain the Minkowski one by analytic continuation only after the computation of the path integral is done. Our result then reproduces the one that we obtained using operator methods quoted in (1.12) above.

We then use the covariant path integral formulation to find the B-dependence of the free energy of the Bosonic string at all genera. The result is that, at a given genus, B^E_{01} appears only in a constraint on the integration over world-sheets in the path integral. At genus g, this constraint restricts the integral over world-sheets to those whose period matrix obeys a condition (see section 4 for notation)

$$\sum_{i=1}^{g} (k_i + \tau_0 m_i)\,\Omega_{ij} - (s_j + \tau_0 n_j) = 0\,, \tag{1.15}$$

where $m_i, n_i, k_i, s_i (i = 1, \ldots, g)$ are 4g integers. This is the condition that the Riemann surface with period matrix Ω_{ij} is the branched cover of a torus with Teichmüller parameter τ_0 identical to that in (1.2). This result, with $B^E = 0$ was found in [1]. The constraint (1.15) restricts the integration over all Riemann surfaces in the path integral computation of the free energy to an integral over those Riemann surfaces which are branched covers of a particular torus. The only place that the constant B-field enters the partition function is in the geometry of the underlying torus.

Note that the two different parameters τ_+ and τ_- defined in (1.13), respectively, have underlying τ-parameters which are obtained by replacing B^E by $-iB$ in τ_0 and $\bar{\tau}_0$, respectively.

We then show that essentially the same constraint occurs for the Green-Schwarz superstring in the same geometrical setting. For the superstring, we present an operator quantization (DLCQ) which gets the genus 1 contribution to the partition function. The essential formula for the genus 1 contribution is

$$\begin{aligned}\frac{F}{V} = &-\sum_{\substack{n=1\\ n\ \mathrm{odd}}}^{\infty}\sum_{k=1}^{\infty}\sum_{s=0}^{k-1}\frac{1}{k^2}\left(\frac{1}{4\pi^2\alpha'\hat{\tau}_2}\right)^5 2^9\left(\theta_4\left(0,2\tau_-\right)\right)^{-8}\left(\bar{\theta}_4\left(0,2\tau_+\right)\right)^{-8}\times\\ &\times e^{-n^2\beta^2/4\pi\alpha'\hat{\tau}_2}\,.\end{aligned} \tag{1.16}$$

An investigation of the superstring at higher genera is not given here. Obtaining the effect of compactification of the null direction on the finite temperature partition function similar to (1.15), but for the super-geometry of the world-sheet should be straightforward.

We then compare our results for the superstring with what we would expect to obtain from the Matrix string model in the same B-field. We show that, if we were to compute the finite temperature partition function of the Matrix string model, we must do a path integral for two-dimensional supersymmetric Yang-Mills theory where the Yang-Mills field variables live on a torus with Teichmüller parameter τ_0. This parameter is identical to the τ_0 which occurred in the string, given in eq. (1.2).

The hypothesis is that the string degrees of freedom are the simultaneous eigenvalues of the matrices. If the matrix elements are doubly periodic functions on the torus, the eigenvalues of the matrices live on branched covers of the torus. We have shown that it is precisely these branched covers which we should expect to become the world-sheets of strings in perturbation theory, where the genus of the branched cover is the genus of the worldsheet. Thus, we can add the conjecture that summing the Matrix string model partition function over the moduli space of branched covers with the appropriate measure should produce string perturbation theory. This has not yet been done in detail beyond genus 1. However, we can check the example of genus 1 explicitly and we find that the partition function of the Matrix string model — where we sum over all genus 1 (unbranched covers) of the basic torus — and the IIA Green-Schwarz superstring are indeed identical. This was done in the absence of B-field in [21]. The extension in this paper to the case with a constant B-field is interesting because the B-field modifies the geometry of both the Matrix model and the quantized string in a simple way. Seeing that this change maps correctly through the limit of the Matrix model which produces perturbative strings is a non-trivial check of the Matrix string model.

A by-product of our analysis is an expression for the Matrix model action in a constant B-field in the $(0,1)$ direction,

$$S = \int dt \left\{ \frac{1}{2g\sqrt{\alpha'}} \mathrm{Tr} \left[\left(D_\tau X^a + B_{01}[X^1, X^a] \right)^2 - \frac{1}{(2\pi\alpha')^2} [X^a, X^b]^2 \right] - \right.$$
$$\left. - \frac{i}{2\pi} \left(\psi^T D_\tau \psi + 2B_{01} \psi^T [X^1, \psi] - \frac{i}{2\pi\alpha'} \psi^T \gamma_a [X^a, \psi] \right) \right\}. \tag{1.17}$$

Finally, we examine the Hagedorn phase transition in a finite B-field. We observe the interesting fact that for the Discrete light-cone quantized closed bosonic string, when one of the directions in which there is electric B-field is compactified, the Hagedorn temperature depends on B:

$$T_H = \sqrt{\frac{1-(2\pi\alpha' B)^2}{16\pi^2\alpha'}}\,.$$

This is remarkable in that it doesn't depend on the compactification radius, so it must hold even if the compactification radius is arbitrarily large. Of course, without the compactification in the first place, T_H would be independent of B and would be the usual closed string value $1/4\pi\sqrt{\alpha'}$. This non-commutativity of compactifying and going to the Hagedorn temperature is a result of the exponential growth of the density of states of the string which is independent of compactification radius. At the Hagedorn temperature, the thermal distribution of string states is unstable and the most favorable configuration is one long string that contains all of the energy. In order to know about the B-field, this long string must wrap the compactified light-like direction. Because of this non-extensive behavior, it always has enough energy to do that, no matter how large the radius R.

In section 2 we shall examine the free energy of a Bosonic string in a background B-field. We will use discrete light-cone quantization (DLCQ) and compute the thermodynamic partition function using operator quantization in the light-cone gauge.

In section 3, we compute the same partition function using the covariant path integral. We do this to illustrate a peculiarity of the compactifications that must be implemented in our computation. It is necessary to do a simultaneous compactification of a null direction in Minkowski space in order to get DLCQ and euclidean time, in order to get a finite temperature path integral. We shall see that the complex identification of time, contained in formulae (3.6) and (3.7) indeed produces a partition function which agrees with the one obtained by operator methods.

In section 4, we use this technique to examine the effect of these simultaneous compactifications on the string path integral at all genera. We show how they lead to a constraint in the path integral measure which restricts the integration over all Riemann surfaces to an integration over the moduli space of branched covers of a particular torus. We find that the external B-field enters the partition function only in these constraints.

In section 5 we extend our genus 1 results for the bosonic string to the type-IIA superstring. We obtain a formula for the genus 1 contribution to the thermodynamic free energy of the DLCQ superstring in a constant B-field quoted in (1.16).

In section 6, we examine the thermodynamic partition function of Matrix string theory in a B-field. We first identify the coupling of a constant B-field to the Matrix model. Then we examine the matrix string free energy in the limit where we keep only the diagonal components of all of the matrix fields. We give an explicit derivation of the thermodynamic partition function at genus 1. The result is identical to the genus 1 superstring partition function that we derived in section 5.

In section 7, we discuss the Hagedorn temperature in the discrete light-cone quantized system. We note a peculiarity of the Hagedorn temperature. It depends on the B-field, but not on the light-cone compactification radius.

In section 8 we give some concluding remarks.

2. Free energy of the Bosonic string in a B-field: DLCQ

2.1 Notation

Let us begin by summarizing some of our notation and conventions. We will use the metric of D-dimensional Minkowski space given by

$$-ds^2 = dx^\mu g_{\mu\nu} dx^\nu \equiv -\left(dx^0\right)^2 + d\vec{x}^{\ 2} = -2dx^+dx^- + \left(d\vec{x}_T\right)^2 , \qquad (2.1)$$

where $\vec{x}$ is the vector made from the $D-1$ spatial components of x^μ, the light-cone coordinates are

$$x^+ = \frac{1}{\sqrt{2}}\left(x^0 + x^1\right), \qquad x^- = \frac{1}{\sqrt{2}}\left(x^0 - x^1\right) \qquad (2.2)$$

and $\vec{x}_T = (x^2, \ldots, x^{D-1})$. We will always consider the string in critical dimensions, so that $D = 26$ for the Bosonic string and $D = 10$ for the superstring.

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In euclidean space

$$X^0 \longrightarrow -iX_E^0\,, \qquad X^i \longrightarrow X^i$$

consequently

$$B_{01} \longrightarrow iB_{01}^E\,.$$

The closed string worldsheet coordinates are denoted by τ and σ where $\sigma \in [0,\pi]$ as in [41]. We define

$$\begin{aligned}\sigma_\pm &= \frac{1}{\sqrt{2}}\,(\tau \pm \sigma)\,,\\ \partial_\pm &= \frac{1}{\sqrt{2}}\,(\partial_\tau \pm \partial_\sigma)\,.\end{aligned}$$

When both the target space and world-sheet are euclidean

$$\tau \longrightarrow -i\sigma_2\,, \qquad \sigma \longrightarrow \sigma_1\,.$$

The four Jacobi theta functions that we will use are defined by

$$\begin{aligned}\theta_1(\nu,\tau) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2} z^{n-1/2}\,, \qquad \theta_3(\nu,\tau) = \sum_{n=-\infty}^{\infty} q^{n^2/2} z^n\,,\\ \theta_2(\nu,\tau) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2} z^{n-1/2}\,, \qquad \theta_4(\nu,\tau) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2} z^n\,,\end{aligned} \tag{2.3}$$

where

$$q = \exp(2\pi i\tau)\,, \qquad z = \exp(2\pi i\nu)\,.$$

We shall also denote by

$$\tilde{\theta}_k(\nu,\tau) = \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \dots$$

the theta function where the $n=0$ term is absent from the sum.

Under the two generators of the $\mathrm{SL}(2,\mathbb{Z})$ modular group, $\tau \to \tau+1$ and $\tau \to -1/\tau$, the modular transformation properties of the theta functions are

$$\begin{aligned}&\theta_1(\nu,\tau+1) = e^{i\pi/4}\theta_1(\nu,\tau)\,, \qquad \theta_2(\nu,\tau+1) = e^{i\pi/4}\theta_2(\nu,\tau)\,,\\ &\theta_3(\nu,\tau+1) = \theta_4(\nu,\tau)\,, \qquad \theta_4(\nu,\tau+1) = \theta_3(\nu,\tau)\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\theta_1\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right) &= -\left(-i\tau\right)^{1/2} e^{\pi i\nu^2/\tau}\theta_1(\nu,\tau)\,,\\ \theta_2\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right) &= -\left(-i\tau\right)^{1/2} e^{\pi i\nu^2/\tau}\theta_4(\nu,\tau)\,,\\ \theta_3\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right) &= -\left(-i\tau\right)^{1/2} e^{\pi i\nu^2/\tau}\theta_3(\nu,\tau)\,,\\ \theta_4\left(\frac{\nu}{\tau},-\frac{1}{\tau}\right) &= -\left(-i\tau\right)^{1/2} e^{\pi i\nu^2/\tau}\theta_2(\nu,\tau)\,.\end{aligned} \tag{2.5}$$

They obey Jacobi's abstruse identity

$$\theta_3^4(0,\tau) - \theta_4^4(0,\tau) - \theta_2^4(0,\tau) = 0\,. \tag{2.6}$$

Jacobi's triple product formulae are

$$\begin{aligned}
\theta_1(\nu,\tau) &= 2q^{1/8}\sin(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)\left(1-2q^n\cos(2\pi\nu)+q^{2n}\right),\\
\theta_2(\nu,\tau) &= 2q^{1/8}\cos(\pi\nu)\prod_{n=1}^{\infty}(1-q^n)\left(1+2q^n\cos(2\pi\nu)+q^{2n}\right),\\
\theta_3(\nu,\tau) &= \prod_{n=1}^{\infty}(1-q^n)\left(1+2q^{n-1/2}\cos(2\pi\nu)+q^{2n-1}\right),\\
\theta_4(\nu,\tau) &= \prod_{n=1}^{\infty}(1-q^n)\left(1-2q^{n-1/2}\cos(2\pi\nu)+q^{2n-1}\right).
\end{aligned} \tag{2.7}$$

The Dedekind eta function is

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \tag{2.8}$$

and has the modular transformation properties

$$\eta(\tau+1) = e^{i\pi/12}\eta(\tau)\,, \qquad \eta\left(-\frac{1}{\tau}\right) = (-i\tau)^{1/2}\eta(\tau)\,. \tag{2.9}$$

We shall also use the notation

$$\bar{\eta}(\tau) = \bar{q}^{1/24}\prod_{n=1}^{\infty}(1-\bar{q}^n)\,, \tag{2.10}$$

where $\bar{q} = e^{-2\pi i\bar{\tau}}$.

The Poisson re-summation formula is

$$\sum_{n=-\infty}^{\infty}\exp\left(-\pi a n^2 - 2\pi i b n\right) = a^{-1/2}\sum_{m=-\infty}^{\infty}\exp\left[-\frac{\pi(m+b)^2}{a}\right]. \tag{2.11}$$

2.2 Solutions of the equations of motion in DLCQ

In DLCQ a null direction is compactified. We shall consider the case where X^+ is identified with $X^+ + 2\pi R$. In closed string theory, the boundary conditions on the worldsheet are periodic,

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma) + \delta^\mu_+ 2\pi R r\,, \tag{2.12}$$

where r is an integer which counts the number of times the string world-sheet wraps the compact direction X^+. When the string wraps the compact direction, the Noether charges are influenced by $B_{\mu\nu}$ as

$$P^\mu = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\partial_\tau X^\mu + 2\pi r B^\mu{}_+ \,. \tag{2.13}$$

We will consider the situation where the only non-zero component of B is B_{+-}. In that case, the only momentum which is influenced by B is P^+ and

$$P^+ = \frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\partial_\tau X^+ + 2\pi B r \,. \tag{2.14}$$

The equations of motion have the light-cone gauge solutions

$$\begin{aligned}
X^-(\tau,\sigma) &= x^- + 2\alpha' P^- \tau \,,\\
X^+(\tau,\sigma) &= x^+ + 2\alpha'(P^+ - 2\pi B r)\tau + 2Rr\sigma + \\
&\quad + \left(\frac{\alpha'}{2}\right)^{(1/2)} i\sum_{n\neq 0}\left(\frac{\alpha_n^+}{n}e^{-2in(\sigma+\tau)} + \frac{\tilde\alpha_n^+}{n}e^{+2in(\sigma-\tau)}\right),\\
\vec X_T(\tau,\sigma) &= \vec x_T + 2\alpha' \vec P_T\tau + \left(\frac{\alpha'}{2}\right)^{(1/2)} i\sum_{n\neq 0}\left(\frac{\vec\alpha_n}{n}e^{-2in(\sigma+\tau)} + \frac{\vec{\tilde\alpha}_n}{n}e^{+2in(\sigma-\tau)}\right),
\end{aligned} \tag{2.15}$$

where $\vec P_T$ and P^- are the total momentum along the transverse and X^- directions, respectively. As we shall show, in the presence of a B-field, p^+ does not coincide with the translation generator along the X^+ direction, which we will denote by P^+.

Since the canonical commutation relations are

$$\left[x^+, P^-\right] = -i$$

and x^+ is a compact variable, the momentum P^- is quantized as

$$P^- = \frac{k}{R}\,,$$

where k is any integer. Substituting the solution (2.15) into the equations of motion, when $k \neq 0$, we get the explicit solutions for the α_n^+ in terms of the transverse oscillators α_n^i

$$\begin{aligned}
\alpha_n^+ &= \frac{1}{2P^-}\left(\frac{2}{\alpha'}\right)^{1/2}\sum_{m=-\infty}^{+\infty} :\ \alpha_{n-m}^i\alpha_m^i : - \frac{1}{P^-}\left(\frac{2}{\alpha'}\right)^{1/2}\delta_{n,0}\,,\\
\tilde\alpha_n^+ &= \frac{1}{2P^-}\left(\frac{2}{\alpha'}\right)^{1/2}\sum_{m=-\infty}^{+\infty} :\tilde\alpha_{n-m}^i\tilde\alpha_m^i :\ - \frac{1}{P^-}\left(\frac{2}{\alpha'}\right)^{1/2}\delta_{n,0}\,,
\end{aligned} \tag{2.16}$$

where $\alpha_0^\mu = p^\mu$. Subtracting these equations for $n = 0$ we get

$$P^- r R = kr = N - \tilde N\,, \tag{2.17}$$

where

$$N = \sum_{n=1}^{\infty} \alpha_{-n}^{i} \alpha_{n}^{i}\,, \qquad \tilde{N} = \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^{i} \tilde{\alpha}_{n}^{i}\,. \tag{2.18}$$

This is the level-matching condition, $L_0 - \tilde{L}_0 = 0$.

Using the solutions (2.16) we get

$$P^+ = \frac{1}{\alpha' P^-}\left(N + \tilde{N} - 2\right) + \frac{1}{2P^-} P^i P^i + \frac{2\pi B}{P^-}\left(N - \tilde{N}\right). \tag{2.19}$$

When we take into account the level matching condition, $N - \tilde{N} = kr$, we see that this agrees with the expression (1.8) which we found by considering a compactified spatial direction and doing an infinite boost in that direction.

The above relation gives the mass-shell condition (1.9).

2.3 Thermodynamic free energy

From ref. [21] it is now easy to construct the free energy of the bosonic string in the presence of a constant B-field. The only difference lies in the mass spectrum operator which is now given by (1.9). We use an expression for the free energy of a single relativistic particle and sum it over the mass spectrum of the string. The result is

$$\begin{aligned} \frac{F}{V} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{k}{\sqrt{2\pi n\beta R}}\right)^{13} \sum_{s=0}^{k-1} \frac{1}{k} \operatorname{Tr}\exp\Bigg(-\frac{n\beta R}{\sqrt{2}}\frac{M^2}{2k} - \frac{nk\beta}{\sqrt{2}R} + \\ + 2\pi i \frac{s}{k}(\tilde{N} - N)\Bigg)\,, \end{aligned} \tag{2.20}$$

where Tr denotes the sum over string states with M^2, N and $\tilde{N}$ the operators given in (1.9) and (2.18). In (2.20), summation over the integer s enforces the level matching condition. The summation over k comes from expanding the free energy of the relativistic particle in a series of exponentials. The summation over n is the sum over light-cone momenta dual to the compact null direction. Some details about how this formula is obtained can be found in [21]. We use the notation for $\tau_{\pm}$, $\hat{\tau}_2$ and ν given in (1.13), (1.14) and (1.3). In terms of these parameters, the free energy of the Bosonic string is given by

$$\frac{F}{V} = -\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{s=0}^{k-1} \frac{1}{k^2}\left(\frac{1}{4\pi^2\alpha'\hat{\tau}_2}\right)^{13} e^{4\pi\hat{\tau}_2 - n^2\beta^2/4\pi\alpha'\hat{\tau}_2} \operatorname{Tr}\left(e^{-2\pi i\bar{\tau}_+ N}\, e^{2\pi i\tau_- \tilde{N}}\right). \tag{2.21}$$

The trace over the string states can be easily computed to give the result of eq. (1.12) which provides the free energy density of the DLCQ bosonic string in a constant B-field. For $B = 0$, eq. (1.12) exactly reproduces the result obtained in ref. [21].

We shall show in the next section how this free energy can be obtained using the covariant Polyakov path integral.

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3. Covariant path integral

In this section we will calculate the free energy of the bosonic string in the presence of a B-field at genus 1 using the covariant approach. We will assume that the metrics of both the world-sheet and the target spacetime have euclidean signatures. The action in euclidean space is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu - 2\pi\alpha' i B^E_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu\right). \tag{3.1}$$

Note that we have also Wick rotated the antisymmetric tensor fields, $B_{01} \to +iB^E_{01}$. We shall show that the compactifications of the target space lead to a well-defined path integral also when B^E is complex. To achieve this result we shall use Seiberg approach to the DLCQ of the bosonic string [29]: we shall obtain a compactified null direction from a compactified spatial direction (given by a small circle of radius R_s) by an infinite boost along the compact direction. At the end of the computation we shall remove the cutoff R_s and we shall analytically continue back to Minkowski B. The cutoff R_s is introduced to show which is the correct prescription to give a path integral representation of the DLCQ free energy. Namely we shall show that the result obtained using Seiberg's approach to DLCQ can be gotten directly compactifying a null direction (i.e. $R_s = 0$) but with a *real* euclidean B^E, which only in the final equation is replaced by a Minkowskian B. This is the procedure we shall adopt in the rest of the paper.

The spacetime is assumed to be flat 26-dimensional euclidean space. The genus 0 Riemann surface is topologically the 2-sphere. Since it has no non-contractible loops, it cannot depend on B^E, β or R.

At genus 1 the world-sheet is a torus. We will take the coordinates of the world-sheet, σ^1 and σ^2 to lie in the range $\sigma^i \in [0, 1)$. Conformal transformations can always be used to put the metric of the torus in the form

$$g_{\alpha\beta} = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \tag{3.2}$$

where then entries are constants given by the complex Teichmüller parameter

$$\tau = \tau_1 + i\tau_2 \,. \tag{3.3}$$

The determinant of the metric is

$$|g| = \tau_2^2 \tag{3.4}$$

and the inverse of the metric is

$$g^{\alpha\beta} = \frac{1}{\tau_2^2} \begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}. \tag{3.5}$$

With this gauge fixing of the metric, the integration over the metric in the Polyakov path integral becomes an integration over the Teichmüller parameter.

We shall also assume that $B^E_{\mu\nu}$ is a constant. The factor i in front of the term containing $B^E_{\mu\nu}$ comes from analytic continuation. The contribution of the B-field term to the world-sheet lagrangian density is proportional to a total derivative and therefore it depends only on the topology of the configuration. We wish to study the situation where the target space has particular compact dimensions. In this case, string configurations which wrap the compact dimensions have non-trivial topology.

Two compactifications will be needed. The first compactifies the light cone Minkowski space by making the identification $\frac{1}{\sqrt{2}}(t+x^1) \sim \frac{1}{\sqrt{2}}(t+x^1) + 2\pi R$. In our euclidean coordinates it becomes the complex identification

$$\left(X^0, X^1, \vec{X}_T\right) \sim \left(X^0 + \sqrt{2}\pi i R, X^1 + \sqrt{2}\pi R, \vec{X}_T\right). \tag{3.6}$$

The second compactification that we shall need is that of euclidean time which is necessary to produce finite temperature $T = 1/\beta$,

$$\left(X^0, X^1, \vec{X}_T\right) \sim \left(X^0 + \beta, X^1, \vec{X}_T\right). \tag{3.7}$$

With these compactifications the only relevant component of the $B^E_{\mu\nu}$ tensor is $B^E_{01} = -B^E_{10} = B^E$.

Following Seiberg [29] we shall consider the light-like compactification as the limit of a compactification on a space-like circle which is almost light-like

$$\left(X^0, X^1, \vec{X}_T\right) \sim \left(X^0 + \sqrt{2}\pi i R, X^1 + \sqrt{2}\pi R\sqrt{1 + 2\left(\frac{R_s}{R}\right)^2}, \vec{X}_T\right) \tag{3.8}$$

with $R_s \ll R$. The light-like circle (3.6) is obtained from (3.8) as $R_s \to 0$. This compactification is related by a large boost with

$$v = \frac{R}{\sqrt{R^2 + 2R_s^2}}$$

to a spatial compactification on

$$\left(X^0, X^1, \vec{X}_T\right) \sim \left(X^0, X^1 + 2\pi R_s, \vec{X}_T\right).$$

The introduction of the cutoff R_s, which will be removed at the end of the calculation, is used to make the analytic continuation of the field B^E to the Minkowski B, well defined at any stage of the computation.

Compactification is implemented by including the possible wrappings of the string world-sheet on the compact dimensions. In general the bosonic coordinates of the string should have a multi-valued part which should take into account the following boundary conditions

$$
\begin{aligned}
X_T^i\left(\sigma_1+1,\sigma_2\right) &= X_T^i\left(\sigma_1,\sigma_2\right), \\
X_T^i\left(\sigma_1,\sigma_2+1\right) &= X_T^i\left(\sigma_1,\sigma_2\right), \\
X^0\left(\sigma_1+1,\sigma_2\right) &= X^0\left(\sigma_1,\sigma_2\right)+\beta m+i\sqrt{2}\pi Rp, \\
X^0\left(\sigma_1,\sigma_2+1\right) &= X^0\left(\sigma_1,\sigma_2\right)+\beta n+i\sqrt{2}\pi Rq, \\
X^1\left(\sigma_1+1,\sigma_2\right) &= X^1\left(\sigma_1,\sigma_2\right)+\sqrt{2}\pi R'p, \\
X^1\left(\sigma_1,\sigma_2+1\right) &= X^1\left(\sigma_1,\sigma_2\right)+\sqrt{2}\pi R'q,
\end{aligned} \tag{3.9}
$$

where $R' = R\sqrt{1+2(R_s/R)^2}$.

As it was shown in refs. [42, 43] with the boundary conditions (3.9) conformal transformations can be used to transform the integration region over the Teichmüller parameter τ into the fundamental domain

$$
\mathcal{F} = \left\{ z\in\mathcal{C}||z|\geq 1, -\frac{1}{2}\leq \mathrm{re}\,z<\frac{1}{2}\right\}. \tag{3.10}
$$

To arrive at the form of the bosonic free energy we obtained in the previous section, eq. (1.12), however, we should use modular transformations to characterize the string wrappings by a single integer [42, 43]. In doing so, the integration domain for the Teichmüller parameters is expanded from $\mathcal{F}$ to the region

$$
\mathcal{S}\equiv\left\{-\frac{1}{2}<\tau_1<\frac{1}{2},\ 0<\tau_2<\infty\right\}. \tag{3.11}
$$

As was shown in the seminal paper by Polchinski [44], this corresponds, on the torus, to consider only the windings of the *time* coordinate of the torus, σ_2, around the target space circle in the X^0 direction. In the Seiberg approach to DLCQ this means boundary conditions of the form

$$
\begin{aligned}
X^0\left(\sigma_1+1,\sigma_2\right) &= X^0\left(\sigma_1,\sigma_2\right)+i\sqrt{2}\pi Rp, \\
X^0\left(\sigma_1,\sigma_2+1\right) &= X^0\left(\sigma_1,\sigma_2\right)+\beta n+i\sqrt{2}\pi Rq, \\
X^1\left(\sigma_1+1,\sigma_2\right) &= X^1\left(\sigma_1,\sigma_2\right)+\sqrt{2}\pi R'p, \\
X^1\left(\sigma_1,\sigma_2+1\right) &= X^1\left(\sigma_1,\sigma_2\right)+\sqrt{2}\pi R'q
\end{aligned} \tag{3.12}
$$

which coincide with (3.9) for $m=0$. We can then write the embedding coordinates X^0, X^1, as a periodic part plus a multi-valued part, according to

$$
\begin{aligned}
X^0 &= X_p^0+n\sigma_2\beta+\sqrt{2}i\pi R\left(p\sigma_1+q\sigma_2\right), \\
X^1 &= X_p^1+\sqrt{2}\pi R'\left(p\sigma_1+q\sigma_2\right).
\end{aligned} \tag{3.13}
$$

With these identifications of the spacetime the action (3.1) can be written as

$$
S=\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{g}g^{\alpha\beta}\partial_\alpha X_p^\mu\partial_\beta X_p^\mu+S_{w.m.}(n,p,q), \tag{3.14}
$$

where we separated the contributions coming from the periodic part from those due to the winding modes. Note that $\vec{X}_{pT} \equiv \vec{X}_T$. Plugging (3.13) into (3.14) for $S_{w.m.}$ one gets

$$\begin{aligned} S_{w.m.}(n,p,q) = & \frac{\beta^2 n^2}{4\pi\alpha'\tau_2} + \frac{\pi R_s^2}{\alpha'\tau_2}\left|p\tau - q\right|^2 + \frac{2\pi i}{\nu\tau_2} \times \\ & \times \left(nq - \left(\tau_1 - 2\pi\alpha' B^E \frac{R'}{R}\tau_2\right) np\right), \end{aligned} \tag{3.15}$$

where $\nu = 4\pi\alpha'/(\sqrt{2}\beta R)$. The part of the path integral which depends on the topological data is gotten by exponentiating the action (3.15) and summing over the integers n, p, q. The expression for the free energy becomes

$$\frac{F}{V} = -\sum_{n,p,q} \int_{\mathcal{S}} d\tau_1 \frac{d\tau_2}{2\tau_2} \frac{\left|\eta\left(\tau\right)\right|^{-48}}{\left(4\pi^2\alpha'\tau_2\right)^{13}} e^{-S_{w.m}(n,p,q)}, \tag{3.16}$$

where the limit $R_s \to 0$ is omitted and the Dedekind eta function and the powers of $4\pi^2\alpha'\tau_2$ arise from the integration over X_p^μ and gauge fixing. The limit $R_s \to 0$ is not defined inside the integral for a complex B^E, in fact if B^E has an imaginary part and we set $R_s = 0$ inside the integral the sums over p and q diverge, since p and q would appear linearly in (3.15). To take this limit, which would provide the DLCQ free energy for the bosonic string, it is convenient to first Poisson re-sum over p and q, then perform the integrals over τ_1 and τ_2 and only at the end remove the cutoff by taking $R_s \to 0$. One gets

$$\sum_{n,p,q} e^{-S_{w.m.}(n,p,q)} = \frac{\alpha'}{R_s^2} \sum_{n,k,s} e^{-S'_{w.m.}(n,k,s)}, \tag{3.17}$$

where

$$S'_{w.m.}(n,k,s) = \frac{\beta^2 n^2}{4\pi\alpha'\tau_2} + \frac{\pi\alpha'}{R_s^2\tau_2}\left[\left((s - \tau_1 k + \frac{\sqrt{2}}{2}B^E\beta R' n\right)^2 + \left(\tau_2 k - \frac{n}{\nu}\right)^2\right]. \tag{3.18}$$

The free energy becomes

$$\begin{aligned} \frac{F}{V} = & -\sum_{n,k,s} \int_{-1/2}^{1/2} d\tau_1 \int_0^\infty \frac{d\tau_2}{2\tau_2} \frac{\left|\eta(\tau)\right|^{-48}}{(4\pi^2\alpha'\tau_2)^{13}} \frac{\alpha'}{R_s^2} \times \\ & \times \exp\left[-\frac{\beta^2 n^2}{4\pi\alpha'\tau_2} - \frac{\pi\alpha'}{R_s^2\tau_2}\left(s - \tau_1 k + \frac{\sqrt{2}}{2}B^E\beta R' n\right)^2 - \frac{\pi\alpha'}{R_s^2\tau_2}\left(\tau_2 k - \frac{n}{\nu}\right)^2\right]. \end{aligned} \tag{3.19}$$

It is not difficult to show that, in the $R_s \to 0$ limit, the $k = 0$ term in this integral is different from zero only for $n = 0$. For $n = 0$ one gets a temperature independent contribution which gives the cosmological constant at one loop [44]. Thus in what follows we can set $n \neq 0$ and $k \neq 0$. In particular, since (3.19) is symmetric for $k \to -k$, $n \to -n$ and $s \to -s$ we can also choose $k = 1, \ldots, \infty$, and multiply by a factor of 2.

To perform the integration over τ_1 it is useful to rewrite the Dedekind eta function in terms of a series as in [41]. One has

$$|\eta(\tau)|^{-48} = e^{4\pi\tau_2} \left| \prod_{m=1}^{\infty} \left(1 - e^{2\pi i\tau m}\right) \right|^{-48} \tag{3.20}$$

and

$$\prod_{m=1}^{\infty} (1 - z^m)^{-24} \equiv \sum_{r=0}^{\infty} p(r) z^r \,, \tag{3.21}$$

where $z = \exp(2\pi i\tau)$. Then the τ_1 dependent terms can be rewritten as

$$\begin{aligned}\int_{-1/2}^{1/2} d\tau_1 \exp \Bigg[-\frac{\pi\alpha'}{R_s^2\tau_2} \left(s - \tau_1 k + \frac{\sqrt{2}}{2} B^E \beta R' n + i \frac{R_s^2\tau_2}{k\alpha'}(r - r') \right)^2 - \\ - \frac{R_s^2\tau_2}{k^2\alpha'}(r - r')^2 + 2\pi i \left(\frac{s}{k} + \frac{\sqrt{2}}{2k} B^E \beta R' n \right)(r - r') \Bigg] .\end{aligned} \tag{3.22}$$

To perform the τ_1 integral it is useful to define in (3.19) new summation variables. Instead of summing over s one can sum over two integers s' and l which are defined according to

$$s' = s + lk \,,$$

where $-\infty < l < \infty$ and $s' = 0, \ldots, k-1$. Introducing the new integration variable

$$\tau_1' = \tau_1 + l$$

eq. (3.22) with the sum over s' and l becomes

$$\begin{aligned}\sum_{l=-\infty}^{\infty} \sum_{s'=0}^{k-1} \int_{-\frac{1}{2}+l}^{\frac{1}{2}+l} d\tau_1' \exp \Bigg[-\frac{\pi\alpha'}{R_s^2\tau_2} \left(s' - \tau_1' k + \frac{\sqrt{2}}{2} B^E \beta R' n + i \frac{R_s^2\tau_2}{k\alpha'}(r - r') \right)^2 - \\ - \frac{R_s^2\tau_2}{k^2\alpha'}(r - r')^2 + 2\pi i \left(\frac{s'}{k} + \frac{\sqrt{2}}{2k} B^E \beta R' n \right)(r - r') \Bigg] .\end{aligned} \tag{3.23}$$

It is now clear that the sum over l can be used to make the τ_1' integral gaussian so that the free energy becomes (with $s' \to s$)

$$\begin{aligned}\frac{F}{V} = -\sum_{n\neq 0} \sum_{k=1}^{\infty} \sum_{s=0}^{k-1} \sum_{r,r'=0}^{\infty} \sqrt{\frac{\alpha'}{k^2 R_s^2}} \int_0^{\infty} \frac{d\tau_2}{\sqrt{\tau_2}(4\pi^2\alpha'\tau_2)^{13}} p(r) p(r') \times \\ \times \exp \Bigg[-\frac{\beta^2 n^2}{4\pi\alpha'\tau_2} + 4\pi\tau_2 - 2\pi\tau_2(r + r') - \frac{R_s^2\tau_2}{k^2\alpha'}(r - r')^2 + \\ + 2\pi i \left(\frac{s}{k} + \frac{\sqrt{2}}{2k} B^E \beta R' n \right)(r - r') - \frac{\pi\alpha'}{R_s^2\tau_2} \left(\tau_2 k - \frac{\sqrt{2}\beta R}{4\pi\alpha'} n \right)^2 \Bigg] .\end{aligned} \tag{3.24}$$

The τ_2 integral can be performed exactly in terms of a Bessel function $K_{25/2}$, but it is much easier to perform at this stage the limit $R_s \to 0$ which is now well defined for any complex B^E. In fact, using the formula

$$\lim_{\epsilon \longrightarrow 0} \frac{1}{\epsilon\sqrt{\pi}} e^{-x^2/\epsilon^2} = \delta(x)$$

with $\epsilon = R_s/\sqrt{\alpha'}$ the free energy reads

$$\frac{F}{V} = -\sum_{n,k=1}^{\infty}\sum_{s=0}^{k-1}\frac{1}{k^2}\int_0^\infty \frac{d\tau_2}{(4\pi^2\alpha'\tau_2)^{13}}\delta\left(\tau_2 - \frac{n}{k\nu}\right)\exp\left[-\frac{\beta^2 n^2}{4\pi\alpha'\tau_2}\right] \times$$
$$\times\left[\eta\left(\frac{s}{k} + i\frac{n}{k\nu}(1 - i2\pi\alpha' B^E)\right)\bar{\eta}\left(\frac{s}{k} + i\frac{n}{k\nu}(1 - i2\pi\alpha' B^E)\right)\right]^{-24}. \quad (3.25)$$

This formula precisely coincides with the one obtained directly in the operatorial light-cone formalism, eq. (1.12), once the euclidean B^E-field is analytically continued to Minkowski space $B^E \to -iB$. Note that this analytical continuation can be performed at any stage of the calculation of this section, thus showing that what we have obtained is a consistent path integral representation of the free-energy of the bosonic string in DLCQ for any complex B. As we shall show in section 5 this path integral representation can also be used for the superstring in the NSR formalism.

The result (3.25) can be obtained more straightforwardly using directly the compactification of the light-like direction x^+, namely with boundary conditions on the embedding coordinates given by (3.12) with $R_s = 0$. When the action $S_{w.m.}$ is exponentiated, since p and q would appear linearly in this case, the summation over p and q leads to a periodic delta function in the path integral measure. This delta function imposes a linear constraint on the Teichmüller parameter. It is given by

$$\sum_{ks}\delta\left(\frac{n\tau_1}{\nu\tau_2} + \frac{2\pi\alpha' B^E n}{\nu} + s\right)\delta\left(\frac{n}{\nu\tau_2} - k\right) = \sum_{ks}\frac{\tau_2}{k^2}\delta^2\left(\tau - \frac{s+\tau_0 n}{k}\right), \quad (3.26)$$

where the sum over integers k and s makes the Dirac delta functions periodic and τ_0 is the parameter defined in (1.2).

Since we have assumed that τ_1 and τ_2 in eq. (3.26), B^E cannot be analytically continued to Minkowski space at this stage. This operation can only be performed at the end of the calculation. The expression for the free energy becomes

$$\frac{F}{V} = -\sum_{n,s,k}\int_{\mathcal{S}} d\tau_1 \frac{d\tau_2}{2k^2}\frac{|\eta(\tau)|^{-48}}{(4\pi^2\alpha'\tau_2)^{13}} e^{-\frac{\beta^2 n^2}{4\pi\alpha'\tau_2}}\delta^2\left(\tau - \frac{s+\tau_0 n}{k}\right). \quad (3.27)$$

It is now easy to perform the integrals in τ_1 and τ_2. After the integrations one can sensibly continue the result to Minkowski space $B^E \to -iB$. The free energy one obtains from (3.27) is identical to (1.12).

The prescription is now clear. One can do any calculation directly using the compactified light-like direction, provided the euclidean B^E field is real. The correct result for the free energy is obtained when the analytic continuation to a Minkowski B-field is performed only at the end of the calculation. That this procedure is correct has been proven by using Seiberg's approach to DLCQ where the analytic continuation to Minkowski B can be done at any stage of the calculation.

As a further application of this procedure we shall now derive the free energy with a Teichmüller parameter defined in the fundamental domain $\mathcal{F}$. The boundary conditions we shall assume on the embedding coordinates are given in (3.9) with $R_s = 0$ so that we can write X^0, X^1, as a periodic part plus a multi-valued part, according to

$$\begin{aligned} X^0 &= X^0_p + (m\sigma_1 + n\sigma_2)\,\beta + \sqrt{2}i\pi R\,(p\sigma_1 + q\sigma_2)\,, \\ X^1 &= X^1_p + \sqrt{2}\pi R\,(p\sigma_1 + q\sigma_2)\,. \end{aligned} \tag{3.28}$$

With (3.28), $S_{w.m.}$ in (3.14) becomes

$$\begin{aligned} S_{w.m.}(m,n,p,q) &= \frac{\beta^2}{4\pi\alpha'\tau_2}\,(m\bar{\tau} - n)\,(\tau m - n) + 2\pi i\frac{\sqrt{2}\beta R}{4\pi\alpha'\tau_2}\,\times \\ &\quad \times \left[nq + |\tau|^2 mp - \tau_1(mq+np)\right] - 2\pi i\sqrt{2}\beta R\frac{B^E}{2}(mq-np)\,. \end{aligned} \tag{3.29}$$

The free energy reads

$$\frac{F}{V} = -\sum_{n,m,p,q}\int_{\mathcal{F}} d\tau_1 \frac{d\tau_2}{2\tau_2}\frac{|\eta\,(\tau)|^{-48}}{(4\pi^2\alpha'\tau_2)^{13}}\,e^{-S_{w.m}(m,n,p,q)}\,, \tag{3.30}$$

where the Teichmüller parameter is integrated over the region $\mathcal{F}$, defined in (3.10). In this expression the sum over m and n is the sum over maps of the torus in which the *space* and *time* time coordinates, respectively, of the torus wrap the target space S^1 m and n times. Since p and p appear linearly in (3.29), the summation over p and q in (3.30) leads to a periodic delta function in the path integral measure. This delta function imposes a linear constraint on the Teichmüller parameter. It is given by

$$\begin{aligned} &\sum_{ks}\delta\left(\frac{|\tau|^2 m - n\tau_1}{\nu\tau_2} + \frac{2\pi\alpha' B^E n}{\nu} + s\right)\delta\left(\frac{n - m\tau_1}{\nu\tau_2} - \frac{2\pi\alpha' B^E m}{\nu} - k\right) = \\ &\quad = \sum_{ks}\frac{\tau_2}{\left[(m2\pi\alpha' B^E/\nu + k)^2 + (m/\nu)^2\right]}\,\delta^2\left(\tau - \frac{s + \tau_0 n}{k + \tau_0 m}\right)\,, \end{aligned} \tag{3.31}$$

where the sum over integers k and s makes the Dirac delta functions periodic. This formula will be generalized to higher genera in the next section. The discrete

Teichmüller parameter then reads

$$\begin{aligned}\tau(n,k,m,s) &= \frac{s+\tau_0 n}{k+\tau_0 m}\,,\\ \tau_1(n,k,m,s) &= \frac{nm+\nu^2(n\phi/2\pi+s)(m\phi/2\pi-k)}{m^2+\nu^2(m\phi/2\pi+k)^2}\,,\\ \tau_2(n,k,m,s) &= \frac{\nu(nk-ms)}{m^2+\nu^2(m\phi/2\pi+k)^2}\,,\end{aligned} \tag{3.32}$$

where

$$\phi = \sqrt{2}\pi\beta R B^E$$

is the flux of the B^E-field on the toroidal space. Integrating over τ_1 and τ_2 the expression for the free energy becomes

$$\begin{aligned}\frac{F}{V} = -\sum_{n,m,k,s} & \frac{1}{[(m\phi/2\pi+k)^2+(m/\nu)^2]}\left(\frac{1}{4\pi^2\alpha'\tau_2}\right)^{13}\times\\ & \times\,\exp\left(-\frac{|m\tau-n|^2\beta^2}{4\pi\alpha'\tau_2}\right)|\eta\,(\tau)|^{-48}\,.\end{aligned} \tag{3.33}$$

Here, the integers n,m,k,s must be chosen so that $\tau(n,k,m,s)$ lies in the fundamental domain $\mathcal{F}$ defined in (3.10).

In eq. (3.33), the periodicity of the free energy when $\phi\to\phi+2\pi\cdot$ integer, is manifest. Moreover when $\phi=2\pi\cdot$ integer the B-field contribution to the partition function can be eliminated. This is consistent with the fact that this particular B^E-field can be eliminated by a gauge transformation.

When $B^E\to -iB$ one obtains from (3.33) the free energy in a physical B-field which is related to (1.12) by a modular transformation.

4. Higher genera

The arguments of section 3 can readily be extended to higher order perturbative contributions to the free energy of the string. These are obtained from the covariant path integral where the worldsheets have higher genus. We shall adopt the procedure described in the previous section namely we shall perform the calculations in DLCQ with a compactified light-like direction and a real euclidean B^E- field and only at the end of the calculation we shall analytically continue the result to Minkowski space.

4.1 Notation

We shall assume that the world-sheet is a Riemann surface Σ_g of genus g. We use complex coordinates

$$z=\sigma_1+i\sigma_2\;,\qquad \bar z=\sigma_1-i\sigma_2\,.$$

Accordingly

$$\partial_1 = \partial_z + \partial_{\bar{z}}\,, \qquad \partial_2 = i\left(\partial_z - \partial_{\bar{z}}\right).$$

A local basis for the differential forms of the second order is given by $d^2\sigma = d\sigma_1 \wedge d\sigma_2$ or $dz \wedge d\bar{z}$. They are related by

$$dz \wedge d\bar{z} = (d\sigma_1 + id\sigma_2) \wedge (d\sigma_1 - id\sigma_2) = -2id\sigma_1 \wedge d\sigma_2 = -2id^2\sigma\,.$$

The homology group of Σ_g, $H_1(\Sigma_g)$, is generated by the homology cycles,

$$a_1, a_2, \ldots, a_g\,, \qquad b_1, b_2, \ldots, b_g$$

which can be chosen so that they have the standard intersection properties

$$a_i \cap a_j = \emptyset\,, \qquad b_i \cap b_j = \emptyset\,, \qquad a_i \cap b_j = \delta_{ij}\,. \tag{4.1}$$

The first cohomology group $H^1(\Sigma_g)$ is generated by g holomorphic differentials

$$\omega_1, \ldots, \omega_g$$

and their conjugates, the anti-holomorphic differentials.

Linear combinations of these can always be chosen so that they obey

$$\oint_{a_i} \omega_j = \delta_{ij}\,, \qquad \oint_{b_i} \omega_j = \Omega_{ij}\,, \tag{4.2}$$

where Ω_{ij} is a $g \times g$ complex, symmetric matrix, called the period matrix of Σ_g. It has positive definite imaginary part.

We shall use the Riemann bilinear relation

$$\int \omega_i \wedge \bar{\omega}_j = \sum_{k=1}^{g} \left(\oint_{a_k} \omega_i \oint_{b_k} \bar{\omega}_j - \oint_{b_k} \omega_i \oint_{a_k} \bar{\omega}_j \right) = -2i\left(\Omega_2\right)_{ij}\,. \tag{4.3}$$

4.2 String action

The string path integral for the vacuum energy is

$$F = -\sum_g g_s^{2g-2} \int [dh_g dX d\Psi] \exp\left[-\frac{1}{4\pi\alpha'} \int_{\Sigma_g} d^2\sigma \left(\sqrt{h} h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu \; - \right.\right. \tag{4.4}$$
$$\left.\left. - \, 2\pi\alpha' i B^E_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \right) \right].$$

The string coupling constant is g_s and its powers weight the genus, $g = 0, 1, \ldots$, of the string's world-sheet. For each value of the genus, g, $[dh_g]$ is an integration measure over all metrics of that genus and is normalized by dividing out the volume of the world-sheet re-parametrization and Weyl groups. We will assume that the metrics of both the world-sheet and the target spacetime have euclidean signature.

We shall consider the situation in which the target space is compactified according to eqs. (3.6) and (3.7). Compactification is implemented by including the possible wrappings of the string worldsheet on the compact dimensions. These form distinct topological sectors in the path integration in (4.4). In the wrapping sectors, the bosonic coordinates of the string should have a multi-valued part which changes by β·integer or $(i)\sqrt{2}R$·integer as it is transported along a homology cycle. This is accomplished by adding a multi-valued classical piece to the embedding coordinate of the string,

$$X = X_{\rm cl} + X_q \,, \tag{4.5}$$

where the quantum part X_q are the variables that are to be integrated in the path integral. The classical part contains the topological part of the configuration. $X_{\rm cl}$ must be harmonic, so that it is a solution of the equation of motion, $\partial\bar{\partial}X = 0$. When $X_{\rm cl}$ is transported around a homology cycle $X_{\rm cl}$ should change by an integer multiples of the compactification circumferences. The integers are the number of times the homology cycle of the worldsheet wraps the compactified spacetime dimension.

It is convenient to work with the differential $dX_{\rm cl}$ which we assume is single-valued. It can be decomposed in complex differentials as

$$dX_{\rm cl} = \partial X_{\rm cl} + \bar{\partial} X_{\rm cl} \,, \tag{4.6}$$

where

$$\partial X_{\rm cl} = \partial_z X_{\rm cl} dz \,, \qquad \bar{\partial} X_{\rm cl} = \partial_{\bar{z}} X_{\rm cl} d\bar{z} \,.$$

Since $X_{\rm cl}$ is harmonic, these can be expanded in the basis for holomorphic and anti-holomorphic differentials, ω_i and $\bar{\omega}_i$, as

$$\partial X_{\rm cl} = \sum_{i=1}^{g} \lambda_i \omega_i \,, \qquad \bar{\partial} X_{\rm cl} = \sum_{i=1}^{g} \bar{\lambda}_i \bar{\omega}_i \tag{4.7}$$

with suitable complex coefficients λ_i. $\bar{\lambda}$ is not necessarily the complex conjugate of λ when there are complex identifications of the embedding variables. In particular, if we choose

$$dX^0_{\rm cl} = \sum_{i=1}^{g} \left(\lambda_i \omega_i + \bar{\lambda}_i \bar{\omega}_i\right) , \qquad dX^1_{\rm cl} = \sum_{i=1}^{g} \left(\gamma_i \omega_i + \bar{\gamma}_i \bar{\omega}_i\right) . \tag{4.8}$$

Then, we require

$$\oint_{a_i} dX^0_{\rm cl} = \beta m_i + \sqrt{2}\pi R i p_i \,, \qquad \oint_{b_i} dX^0_{\rm cl} = \beta n_i + \sqrt{2}\pi R i q_i \,,$$
$$\oint_{a_i} dX^1_{\rm cl} = \sqrt{2}\pi R p_i \,, \qquad \oint_{b_i} dX^1_{\rm cl} = \sqrt{2}\pi R q_i \,.$$

With (4.2) and (4.3) we use these equations to solve for the constants in (4.8). We find

$$
\begin{aligned}
\lambda &= \frac{i}{2}\Omega_2^{-1}\left[\bar{\Omega}(\beta m + i\sqrt{2}\pi Rp) - (\beta n + i\sqrt{2}\pi Rq)\right], \\
\bar{\lambda} &= -\frac{i}{2}\Omega_2^{-1}\left[\Omega(\beta m + i\sqrt{2}\pi Rp) - (\beta n + i\sqrt{2}\pi Rq)\right], \\
\gamma &= \frac{i\sqrt{2}\pi R}{2}\Omega_2^{-1}\left(\bar{\Omega}p - q\right), \\
\bar{\gamma} &= -\frac{i\sqrt{2}\pi R}{2}\Omega_2^{-1}\left(\Omega p - q\right).
\end{aligned} \tag{4.9}
$$

Where we have used an obvious vector notation for multiplying the period matrix and integer-valued vectors p_i and q_i, etc.

The string action for the classical configurations

$$
\begin{aligned}
S_{\rm cl} &= \frac{1}{4\pi\alpha'}\int_{\Sigma_g} d^2\sigma\left(\partial_\alpha X^\mu_{\rm cl}\partial_\alpha X^\mu_{\rm cl} - 2\pi\alpha' i B^E_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu_{\rm cl}\partial_\beta X^\nu_{\rm cl}\right) \\
&= \frac{i}{2\pi\alpha'}\int_{\Sigma_g}\left(\partial X^\mu_{\rm cl}\wedge\bar{\partial}X^\mu_{\rm cl} - 2\pi\alpha' B^E_{\mu\nu}\partial X^\mu_{\rm cl}\wedge\bar{\partial}X^\nu_{\rm cl}\right)
\end{aligned} \tag{4.10}
$$

using (4.3) and (4.8), becomes

$$
S_{\rm cl} = \frac{1}{\pi\alpha'}\left[\lambda\Omega_2\bar{\lambda} + \gamma\Omega_2\bar{\gamma} + 2\pi\alpha' B^E\left(\lambda\Omega_2\bar{\gamma} - \gamma\Omega_2\bar{\lambda}\right)\right], \tag{4.11}
$$

where, as usual, $B^E = B^E_{01}$. The part of the string action which contains the winding integers, can then be obtained by the solutions (4.9) and reads

$$
\begin{aligned}
S = {} & \frac{\beta^2}{4\pi\alpha'}\left(m\Omega^\dagger - n\right)\Omega_2^{-1}\left(\Omega m - n\right) + 2\pi i\frac{\sqrt{2}\beta R}{4\pi\alpha'}\frac{1}{2}\times \\
& \times\left[\left(p\Omega^\dagger - q\right)\Omega_2^{-1}\left(\Omega m - n\right) + \left(m\Omega^\dagger - n\right)\Omega_2^{-1}\left(\Omega p - q\right) - 4\pi\alpha' B^E(mq - np)\right].
\end{aligned} \tag{4.12}
$$

As expected the B-part of the action does not depend on the period matrix and it is a purely topological term. Note that the integers p_i and q_i appear linearly in a purely imaginary term in the action. When the action is exponentiated and summed over p_i and q_i, the result will be periodic Dirac delta functions. It can be shown that these delta functions impose a linear constraint on the period matrix of the world-sheet. Thus, with the appropriate jacobian factor, the net effect is to insert into the path integral measure the following expression,

$$
\begin{aligned}
& \sum_{mnks} e^{-\frac{\beta^2}{4\pi\alpha'}\left(m\Omega^\dagger - n\right)\Omega_2^{-1}\left(\Omega m - n\right)}\left(\frac{\nu^2}{1 + (2\pi\alpha' B^E)^2}\right)^g |\det\Omega_2| \\
& \prod_{j=1}^g \delta\left(\sum_{i=1}^g (k_i + \tau_0 m_i)\,\Omega_{ij} - (s_j + \tau_0 n_j)\right),
\end{aligned} \tag{4.13}
$$

where τ_0 is defined in (1.2) and $\nu = 4\pi\alpha'/\sqrt{2}\beta R$ as in (1.3). Consequently, the integration over metrics in the string path integral is restricted to those for which the period matrix obeys the constraint

$$\sum_{i=1}^{g} (k_i + \tau_0 m_i)\, \Omega_{ij} - (s_j + \tau_0 n_j) = 0 \tag{4.14}$$

for all combinations of the $4g$ integers m_i, n_i, k_i, s_i such that Ω_{ij} is in a fundamental domain $\mathcal{F}$ of the Riemann surface. For genus 1 this equation provides the discrete Teichmüller parameter defined in (3.32).

The Riemann surfaces with the constraint (4.14) have been classified in ref. [1]. They are the branched coverings of the torus with Teichmüller parameter τ_0.

5. Superstring

We shall now consider the DLCQ of a closed superstring in ten dimensions in the presence of a B-field. In this section we shall derive, following Atick and Witten [38], the expression of the free energy using the path integral for the superstring in the Neveu-Schwarz-Ramond (NSR) formalism. By means of modular invariance [42, 43], we shall then obtain the form of the superstring free energy that has to be compared with the one of the matrix string theory.

The strategy we shall adopt is the one described in section 3, namely we will use a compactified light-like direction X^+ and a real euclidean B^E field and only at the end of the calculation we shall analytically continue the result to Minkowski space $B^E \to -iB$.

The superstring covariant action in the NSR formalism in the presence of an antisymmetric tensor $B_{\mu\nu}$ is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{g} g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu - i2\pi\alpha' B^E_{\mu\nu} \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu - \right. \\ \left. - 2i\alpha' e \bar{\psi}^\mu \rho^a e^\alpha_a \partial_\alpha \psi_\mu - \pi (2\alpha')^2 B^E_{\mu\nu} \epsilon^{\alpha\beta} \bar{\psi}^\mu \rho_a e^a_\alpha \partial_\beta \psi^\nu \right). \tag{5.1}$$

Here ψ^μ is a Majorana spinor, $g_{\alpha\beta}$ is the world-sheet metric, e^a_α the corresponding *zweibein* and e its determinant. At one loop, i.e. on a torus, we shall take for $g_{\alpha\beta}$ the metric given in (3.2). To provide the free energy, this action has to be inserted in the string path integral, the metrics of both the world sheet and the target spacetime having euclidean signature.

For a constant B-field, the contribution of the B-field term to the world-sheet lagrangian density is proportional to a total derivative. Since the boundary conditions on the fermions are periodic (Ramond sector) or anti-periodic (Neveu-Schwarz sector) the coupling of B to the ψ^μ in (5.1) actually vanishes. Thus the only B-dependent contribution comes from the bosonic part of the action.

The boundary conditions we shall consider on the bosonic coordinates are given by eqs. (3.9) with $R_s = 0$. The compactification of the coordinate X^0, with the appropriate modification of the GSO projection to make space-time fermions anti-periodic, introduces the temperature, so that the path integral with the action (5.1) computes the thermodynamic free energy.

The part of the string action which contains the winding integers, $S_{w.m.}$ is identical to that of the bosonic string (3.29). Note that the integers p and q appear linearly in a purely imaginary term. Furthermore, since they will come from the compactification of the light-cone direction, this is the only place that they will appear in the string path integral. Unlike m and n which appear in the weights of the sum over spin structures, the compactification of the light-cone direction does not modify the GSO projection. Thus, when the action is exponentiated and summed over p and q the result will be, as in the bosonic case, periodic delta functions.

The type-II superstring free energy in the Neveu-Schwarz-Ramond formulation of superstring in a B-field background can then be obtained as in [38]. It reads

$$\begin{aligned}\frac{F}{V} = & -\frac{1}{4}\left(\frac{1}{4\pi^2\alpha'}\right)^5 \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^6}\left|\frac{1}{\eta(\tau)}\right|^{24} \sum_{n,m,p,q=-\infty}^{+\infty} e^{-S_{w.m.}(m,n,p,q)} \times \\ & \times \left[\left(\theta_2^4\bar\theta_2^4 + \theta_3^4\bar\theta_3^4 + \theta_4^4\bar\theta_4^4\right)(0,\tau) + e^{i\pi(n+m)}\left(\theta_2^4\bar\theta_4^4 + \theta_4^4\bar\theta_2^4\right)(0,\tau) - \right. \\ & \left. - e^{i\pi n}\left(\theta_2^4\bar\theta_3^4 + \theta_3^4\bar\theta_2^4\right)(0,\tau) - e^{i\pi m}\left(\theta_3^4\bar\theta_4^4 + \theta_4^4\bar\theta_3^4\right)(0,\tau)\right], \end{aligned} \tag{5.2}$$

where the Teichmüller parameters are integrated over the region $\mathcal{F}$, which is the fundamental domain of the torus. Characterizing these wrappings by a single integer [42, 43], the integration domain for the Teichmüller parameters is expanded from $\mathcal{F}$ to the region $\mathcal{S}$ defined in (3.11).

If we set $m = 0$ and integrate over the region $\mathcal{S}$ in (5.2) we get [21]

$$\begin{aligned}\frac{F}{V} = & -\left(\frac{1}{4\pi^2\alpha'}\right)^5 \int_{-1/2}^{1/2} d\tau_1 \int_0^\infty \frac{d\tau_2}{\tau_2^5}\left|\frac{1}{\eta(\tau)}\right|^{24} 2\sum_{\substack{n=1\\ n\ odd}}^{+\infty}\sum_{k=1}^{+\infty}\sum_{s=0}^{k-1}\frac{1}{k^2}e^{-\frac{\beta^2 n^2}{4\pi\alpha'\tau_2}} \times \\ & \times \left|\theta_2(0,\tau)\right|^8 \delta\left(\tau_1 + \frac{2\pi\alpha' B^E n}{\nu k} - \frac{s}{k}\right)\delta\left(\tau_2 - \frac{n}{\nu k}\right). \end{aligned} \tag{5.3}$$

The product representation of $\theta_2(0,\tau)$ in (2.7) implies that

$$\left|\frac{1}{\eta(\tau)}\right|^{24}\left|\theta_2(0,\tau)\right|^8 = 2^8\prod_{n=1}^{\infty}\left|\frac{1+e^{2\pi i n\tau}}{1-e^{2\pi i n\tau}}\right|^{16} \equiv 2^8\left|\theta_4(0,2\tau)\right|^{-16}. \tag{5.4}$$

When the B^E field is continued to Minkowski space $B^E \to -iB$. the free energy for the type-II superstring is given in (1.16). Equation (1.16) gives the free energy of the DLCQ type-II superstring in a constant B-field which, in the next sections, will be compared with the one derived for the matrix string theory.

6. The thermodynamic partition function of the matrix string theory in a B-field

In this section we shall first find the Matrix string theory action in the presence of a background B-field whose only non-zero component is B_{01}. Then we will consider the resulting Matrix string theory in the limit which should coincide with free type-IIA strings. We shall show that, when the thermodynamic partition function of Matrix string theory in this limit is computed, it coincides with the one obtained in the previous section for the DLCQ type-II superstring.

The matrix model of M-theory is the effective field theory of the low energy interactions of a gas of D0-branes propagating on ten-dimensional Minkowski space. When one spatial direction is compactified, the matrix model which describes the resulting theory is $1+1$-dimensional maximally supersymmetric Yang-Mills theory. This theory contains T-duality in a manifest way - it can either describe D0-branes or their T-dual, D1-branes which wrap the compact direction where the compactification radius R_1 should be replaced by the dual radius α'/R_1. To find the coupling of a background B-field to D0-branes on Minkowski space with a compact dimension, it is convenient to begin with the T-dual of that situation which is wrapped D1-branes, no B-field, and space-time with the metric (1.10).

The $1+1$-dimensional Yang-Mills theory which describes this system is gotten by starting with ten-dimensional $\mathrm{U}(N)$ super Yang-Mills theory in the background metric (1.10) [45]. In euclidean space time, the ten-dimensional action is

$$S=\frac{1}{2}\int d^{10}\xi\,\mathrm{Tr}\sqrt{G}\left(F_{\mu\nu}G^{\mu\rho}G^{\nu\sigma}F_{\rho\sigma}-\frac{i}{\pi}\,\mathrm{Tr}\,\bar{\psi}\gamma^a V_a^{\mu}D_\mu\psi\right), \qquad (6.1)$$

where the fields A_μ and ψ are both in the adjoint representation of $\mathrm{U}(N)$. The covariant derivative is given by

$$D_\mu=\partial_\mu-ig_{\mathrm{YM}}\left[A_\mu,\ldots\right]$$

$G_{\mu\nu}$ is a metric and V_μ^a the corresponding *zehnbein*. In euclidean space, in terms of the euclidean B^E-field the metric reads

$$G^{\mu\nu}=\begin{pmatrix}1+(2\pi\alpha' B^E)^2 & -2\pi\alpha' B^E & 0 & \ldots\\ -2\pi\alpha' B^E & 1 & 0 & \ldots\\ 0 & 0 & 1 & \ldots\\ \ldots & \ldots & \ldots & \ldots\end{pmatrix}. \qquad (6.2)$$

Ten-dimensional Yang-Mills theory is reduced to two-dimensional Yang-Mills theory by assuming that the fields are independent on the coordinates $2,\ldots,9$. In 1+1 dimensions there is a gauge field A_α, $\alpha=1,2$, 8 adjoint scalars X^i, $i=2,\ldots,9$ and a 16-component Majorana spinor which can by arranged into 8 two-dimensional

Majorana spinors. The action (6.1) becomes

$$S=\frac{1}{2}\int d^2\sigma\left[\sqrt{g}g^{\alpha\beta}D_\alpha X^i D_\beta X^i - g_{\rm YM}^2\sum_{i<j}\left[X^i,X^j\right]^2+\sqrt{g}F_{\alpha\beta}g^{\alpha\gamma}g^{\beta\delta}F_{\gamma\delta}\right.$$
$$\left.-\frac{i}{\pi}\left(\sqrt{g}\psi^T\gamma^a e_a^\alpha D_\alpha\psi+g_{\rm YM}\psi^T\gamma_i[X^i,\psi]\right)\right], \tag{6.3}$$

where $g_{\rm YM}$ is the ten-dimensional Yang-Mills coupling constant, $g^{\alpha\beta}$ is the $(0,1)$ part of the metric (6.2) and e_a^α the corresponding *zweibein*. This is the action describing the low energy dynamics of N coincident D1-brane in the background metric $g_{\alpha\beta}$. It can be easily seen that the linear terms in B in (6.3) are exactly those found in [33, 34]. When the direction 1 is compactified, using the Buscher rules for T-duality, (6.3) is equivalent to D0-branes on a space with the same compactified direction, the euclidean metric and a non-zero B-field in the $(0,1)$ direction.

To arrive at the matrix string point of view [13], (which we shall refer to as DVV) we have to take into account that, conventionally, the matrix theory is related to M-theory via the compactification of the 11th direction on a circle of radius R_{11}. We then re-scale the field variables, the coupling constant and the coordinates in (6.3) in such a way to introduce the compactification radius R_{11}, the string tension $1/2\pi\alpha'$ and coupling constant g_s. The necessary re-scalings are

$$\begin{aligned}
&X^i\longrightarrow\left(\frac{1}{2\pi\alpha'}\right)^{1/2}X^i\,, &&\psi\longrightarrow\left(\frac{R_{11}^2}{2\pi\alpha'}\right)^{1/4}\psi\,,\\
&A^1\longrightarrow\left(\frac{R_{11}^2}{2\pi\alpha'}\right)^{1/2}A^1\,, &&A^2\longrightarrow(2\pi\alpha')^{1/2}A^2\,, \qquad g_{\rm YM}\longrightarrow\frac{R_{11}}{g_s\sqrt{\alpha'}}\,,\\
&\sigma^1\longrightarrow\left(\frac{2\pi\alpha'}{R_{11}^2}\right)^{1/2}\sigma^1\,, &&\sigma^2\longrightarrow\left(\frac{1}{2\pi\alpha'}\right)^{1/2}\sigma^2\,.
\end{aligned}\tag{6.4}$$

The action (6.3) then becomes

$$S_{\rm DVV}=\int d^2\sigma\left\{\frac{1}{2R_{11}}\left[\left(D_2X^i-R_{11}B^ED_1X^i\right)^2+\left(\frac{R_{11}}{2\pi\alpha'}D_1X^i\right)^2+g_s^2\alpha' F_{\mu\nu}^2\;-\right.\right.$$
$$\left.-\;\frac{R_{11}^2}{4\pi^2\alpha'^3g_s^2}\sum_{i<j}[X^i,X^j]^2\right]-$$
$$-\frac{iR_{11}}{2\pi}\psi^T\left(\frac{1}{R_{11}}D_2\psi+\frac{1}{2\pi\alpha'}\gamma D_1\psi-B^ED_1\psi-\right.$$
$$\left.\left.-\;\frac{i}{2\pi\alpha'^{3/2}g_s}\gamma_i[X^i,\psi]\right)\right\}. \tag{6.5}$$

The two-dimensional world-sheet here is a cylinder with compact spatial coordinate $\sigma_1\in[0,1)$ and euclidean time $\sigma^2\in(-\infty,\infty)$. Later, when we consider the euclidean action in the partition function, euclidean time will also be compact. This is the action that, according to the DVV construction [13], should describe the non-perturbative particle spectrum of the type-IIA string in a B-field in terms of the states that can be made up from infinitely many D0-branes. The compactification on the circle S^1 along the 1-direction and the reinterpretation of the X^1 as a covariant derivative D_1 with a gauge field defined on S^1, amounts to applying a T-duality transformation along the S^1 direction, thereby turning the D0-branes into D1-branes. This is the reason why this action can be obtained from the D1-brane action (6.3).

The hamiltonian corresponding to the action (6.5) and that will be used in the calculation of the thermodynamic partition function, reads

$$\begin{aligned} H = \frac{R_{11}}{2}\int_0^1 d\sigma_1 \operatorname{Tr}\Big[&\Pi_i^2 + 2iB^E\Pi^i D_1X^i + \frac{1}{4\pi^2\alpha'^2}(D_1X^i)^2 - \\ &- \frac{1}{4\pi^2\alpha'^3 g_s^2}\sum_{i<j}\left[X^i, X^j\right]^2 + \frac{E^2}{g_s^2\alpha'} + \frac{i}{\pi}B^E\psi^T\cdot D_1\psi - \\ &- \frac{i}{2\pi^2\alpha'}\psi^T\gamma\cdot D_1\psi - \frac{1}{2\pi^2\alpha'^{3/2}g_s}\psi^T\gamma_i\left[X^i,\psi\right]\Big]. \end{aligned} \tag{6.6}$$

The mass-dimensions of H and of the fields are

$$[H] = M\,, \qquad [X^a] = M^{-1}\,, \qquad [\psi] = M^0\,, \qquad [A^{1,2}] = M^0\,.$$

R_{11} is the radius of the dimension that must be compactified in order to obtain the matrix description of M-theory — in its original form the Matrix model describes M-theory in a reference frame which has infinite momentum in the 11th direction. In the more sophisticated proposal of ref. [3], this compactified direction is the light-cone direction X^+. It was shown by Seiberg [29] that, with certain assumptions, a boost to the frame with infinite momentum of a theory compactified on a small circle R_{11} is equivalent to one with the light-cone compactified. For a more recent discussion of these issues see [46, 47, 48]. Here, since we are actually describing DLCQ M-theory, we shall make the replacement

$$R_{11} \longrightarrow R\,.$$

The hypothesis is that this model describes DLCQ M-theory with one dimension compactified (to describe the DLCQ IIA superstring). The canonical momenta which appear in (6.6) are normalized so that they have the conventional canonical commutators,

$$\begin{aligned} \left[X^i_{ab}(\sigma), \Pi^j_{cd}(\sigma')\right] &= i\delta_{ad}\delta_{bc}\delta^{ij}\delta(\sigma-\sigma')\,, \\ [A_{ab}(\sigma), E_{cd}(\sigma')] &= i\delta_{ad}\delta_{bc}\delta(\sigma-\sigma')\,, \\ \{\psi_{ab}(\sigma), \psi_{cd}(\sigma')\} &= \frac{1}{2\pi}\delta_{ad}\delta_{bc}\delta(\sigma-\sigma')\,. \end{aligned} \tag{6.7}$$

Note that $g_s^2\alpha'$ plays the role of the inverse square of the Yang-Mills coupling constant in 1+1 dimension: $g_s^2\alpha' = g_{\rm YM}^{-2}$

6.1 Matrix theory in a B-field

In this subsection, we will find the matrix model which describes M-theory in a B-field on an eleven-dimensional space whose only compactification is the null direction X^+. To do this, we shall follow an inverse DVV procedure [13]. This result will be achieved by first decompactifying the T-dual circle of radius R_1 and then interchanging the role of the 11th and the 1st directions.

To this purpose we first rewrite the hamiltonian (6.6) in terms of the 11-dimensional Plank length l_p, the radius of the compactified direction, R_1, and the string coupling constant g_s. The relation between these constants and the string scale $\sqrt{\alpha'}$ is defined as

$$g_s\sqrt{\alpha'} = R_1\,, \qquad g_s^{1/3}\sqrt{\alpha'} = l_p\,. \tag{6.8}$$

This leads to the hamiltonian

$$\begin{aligned} H = \frac{R_{11}}{2}\int_0^1 d\sigma_1\,\mathrm{Tr}\Bigg[&\Pi_i^2 + \frac{R_1^2}{4\pi^2 l_p^6}(D_1X^i)^2 - \frac{1}{4\pi^2 l_p^6}\sum_{i<j}\left[X^i,X^j\right]^2 + 2iB^E D_1X^i\Pi^i + \\ &+\frac{E^2}{R_1^2} + \frac{i}{\pi}B^E\psi^T D_1\psi - \frac{1}{2\pi^2 l_p^3}\psi^T\gamma_i\left[X^i,\psi\right] - \\ &-\frac{iR_1}{2\pi^2 l_p^3}\psi^T\gamma\cdot D_1\psi\Bigg]\,, \end{aligned} \tag{6.9}$$

where $i = 1,\ldots,8$ now labels the transverse coordinates. We now decompactify the 1st direction, so that, after the usual T-duality, we can identify the covariant derivative D_1 with the X^1 coordinate, according to

$$iR_1D_1 \longrightarrow X^1\,. \tag{6.10}$$

With this procedure the hamiltonian (6.9) becomes

$$\begin{aligned} H = \frac{R_{11}}{2}\,\mathrm{Tr}\Bigg[&\Pi_a^2 + 2\frac{B^E\alpha'}{l_p^3}[X^1,X^a]\Pi^a - \frac{1}{4\pi^2 l_p^6}[X^a,X^b]^2 - \\ &-\frac{1}{2\pi^2 l_p^3}\psi^T\gamma_a[X^a,\psi] + \frac{B^E\alpha'}{\pi l_p^3}\psi^T[X^1,\psi]\Bigg]\,, \end{aligned} \tag{6.11}$$

where $a = 1,\ldots,9$. We now interchange the 11th and the 1st directions by introducing eleven-dimensional Planck units: $R_{11} = g\sqrt{\alpha'}$ and $l_p^2 = \alpha' g^{2/3}$. The 1-11 flip, has been shown to correspond to a chain of $T - S - T$ duality in [33]. With this substitution we obtain the final result for the hamiltonian of the Matrix model in a constant B-field in the $(0,1)$ direction

$$\begin{aligned} H = &\frac{1}{2g\sqrt{\alpha'}}\,\mathrm{Tr}\left[g^2\alpha'\Pi_a^2 + 2B^E g\sqrt{\alpha'}\Pi^a[X^1,X^a] - \frac{1}{(2\pi\alpha')^2}[X^a,X^b]^2\right] - \\ &-\frac{1}{2\pi}\,\mathrm{Tr}\left[\frac{1}{2\pi\alpha'}\psi^T\gamma_a[X^a,\psi] - 2B^E\psi^T[X^1,\psi]\right]\,. \end{aligned} \tag{6.12}$$

When the σ_2 coordinate is identified with the euclidean time τ the Matrix model action then reads

$$S = \int dt \left\{ \frac{1}{2g\sqrt{\alpha'}} \operatorname{Tr} \left[\left(D_\tau X^a + iB^E[X^1, X^a]\right)^2 - \frac{1}{(2\pi\alpha')^2}[X^a, X^b]^2 \right] - \right.$$
$$\left. - \frac{i}{2\pi} \left(\psi^T D_\tau \psi + 2iB^E \psi^T [X^1, \psi] - \frac{i}{2\pi\alpha'} \psi^T \gamma_a [X^a, \psi] \right) \right\}. \qquad (6.13)$$

Each of the adjoint scalar matrices X^a is a hermitean $N \times N$ matrix, where N is the number of 0-branes and the relation between the coupling constant and the compactification radius of the 11th direction $g\sqrt{\alpha'} = R_{11}$, has been used. This action describes a stack of N D0-branes in the presence of a B-field in the $(0, 1)$ direction.

Since (6.13) is the action for N coincident D0-branes, it should be possible to derive it also from dimensional reduction of the ten-dimensional U(N) super Yang-Mills theory (6.1). Namely we can arrive at (6.13) by a further dimensional reduction of (6.3) assuming that all fields are independent of the coordinate 1, calling time the coordinate 2 and performing the following re-scalings of the fields, coupling constant and coordinate

$$X^a \longrightarrow \left(\frac{1}{2\pi\alpha' g_{\rm YM}}\right)^{1/2} X^a\,, \qquad A^2 \longrightarrow (2\pi\alpha' g_{\rm YM})^{1/2} A^2\,,$$
$$\sigma^2 \longrightarrow \left(\frac{1}{2\pi\alpha' g_{\rm YM}}\right)^{1/2} \tau\,, \qquad g_{\rm YM} \longrightarrow \frac{g^2}{2\pi}\,. \qquad (6.14)$$

6.2 Matrix string free energy

Using the hamiltonian (6.6) we shall now construct the thermodynamic partition function Z of matrix string theory in the limit of vanishing string coupling, $g_s \to 0$. In DLCQ Z is given by (1.1) where H is the matrix string theory hamiltonian in (6.6). The trace of $\exp(-\beta H/\sqrt{2})$ is to be taken over gauge invariant states of the two-dimensional super-Yang-Mills theory. This trace has the standard euclidean field theory path integral expression [49]

$$Z[\beta] = \sum_{N=0}^{\infty} \int [dA_\alpha][dX^i][d\psi^a] \exp\left(-\frac{\beta N}{\sqrt{2}R} - S_E[A, X^i, \psi]\right), \qquad (6.15)$$

where S_E is the euclidean action

$$S_E = \frac{1}{4\pi\alpha'} \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \operatorname{Tr} \times$$
$$\times \left\{ \frac{1 + (2\pi\alpha' B^E)^2}{\nu} D_1 X^i D_1 X^i + \nu D_2 X^i D_2 X^i - 4\pi\alpha' B^E D_1 X^i D_2 X^i + \right.$$
$$+ \nu g_s^2 \alpha' F_{\mu\nu}^2 - \frac{1}{\nu\alpha' g_s^2} \sum_{i<j} \left[X^i, X^j\right]^2 i2\alpha' \times \qquad (6.16)$$
$$\left. \times \left(\psi^T D_2 \psi + \frac{1}{\nu} \psi^T \gamma D_1 \psi - \frac{2\pi\alpha' B^E}{\nu} \psi^T D_1 \psi - \frac{i}{\nu g_s \sqrt{\alpha'}} \psi^T \gamma^i \left[X^i, \psi\right] \right) \right\}.$$

Here we have rescaled the time σ_2 so that the integration is over a box of area one, $0 \leq \sigma_\mu < 1$, and β appears as a factor in various coupling constants. Boundary conditions in the path integral are

$$\begin{aligned}
A_\mu(\sigma_1+1,\sigma_2) &= A_\mu(\sigma_1,\sigma_2)\,,\\
X^i(\sigma_1+1,\sigma_2) &= X^i(\sigma_1,\sigma_2)\,,\\
\psi(\sigma_1+1,\sigma_2) &= \psi(\sigma_1,\sigma_2)\,,\\
A_\mu(\sigma_1,\sigma_2+1) &= A_\mu(\sigma_1,\sigma_2)\,,\\
X^i(\sigma_1,\sigma_2+1) &= X^i(\sigma_1,\sigma_2)\,,\\
\psi(\sigma_1,\sigma_2+1) &= -\psi(\sigma_1,\sigma_2)\,.
\end{aligned} \tag{6.17}$$

The anti-periodicity of the fermion field in the euclidean time comes from taking the trace in (1.1).

The limit $g_s \to 0$ of this theory was formulated by DVV [13]. Their hypothesis is that, in this limit, the field configurations which have finite action are simultaneously diagonalizable matrices. They can therefore all be related to diagonal field configurations by a gauge transformation

$$\begin{aligned}
X^i(\sigma) &= \mathrm{U}(\sigma)X^i_D(\sigma)U^{-1}(\sigma)\,,\\
\psi(\sigma) &= \mathrm{U}(\sigma)\psi_D(\sigma)U^{-1}(\sigma)\,,\\
A_\mu(\sigma) &= iU(\sigma)\left(\partial_\mu - iA^D_\mu(\sigma)\right)U^{-1}(\sigma)
\end{aligned} \tag{6.18}$$

X^i_D, ψ_D and A^D_μ are diagonal matrices. The fields X^i, ψ and A_μ have the (anti-)periodic boundary conditions (6.17). The spectrum of (anti-)periodic matrices must be (anti-)periodic. However, the individual eigenvalues need not be. They can be periodic up to permutations (and gauge transformations generated by the Cartan sub-algebra),

$$\begin{aligned}
(A^D_\mu)_a(\sigma_1+1,\sigma_2) &= (A^D_\mu)_{P(a)}(\sigma_1,\sigma_2) + 2\pi(n_\mu)_a + \partial_\mu\theta_a\,,\\
(X^i_D)_a(\sigma_1+1,\sigma_2) &= (X^i_D)_{P(a)}(\sigma_1,\sigma_2)\,,\\
(\psi_D)_a(\sigma_1+1,\sigma_2) &= (\psi_D)_{P(a)}(\sigma_1,\sigma_2)\,,\\
(A^D_\mu)_a(\sigma_1,\sigma_2+1) &= (A^D_\mu)_{Q(a)}(\sigma_1,\sigma_2) + 2\pi(m_\mu)_a + \partial_\mu\phi_a\,,\\
(X^i_D)_a(\sigma_1,\sigma_2+1) &= (X^i_D)_{Q(a)}(\sigma_1,\sigma_2)\,,\\
(\psi_D)_a(\sigma_1,\sigma_2+1) &= -(\psi_D)_{Q(a)}(\sigma_1,\sigma_2)\,,
\end{aligned} \tag{6.19}$$

where $a = 1,\ldots,N$, $P(a)$ and $Q(a)$ are permutations and n_μ, m_μ are integers. Consistency requires that the two permutations commute,

$$PQ = QP\,. \tag{6.20}$$

To compute the partition function, in the $g_s \to 0$ limit, we should now do the path integral (6.15) over only the diagonal components of the matrix fields with the action

$$\begin{aligned} S_{\rm diag} = \frac{1}{4\pi\alpha'} \int_0^1 d\sigma_1 \int_0^1 d\sigma_2 \,{\rm Tr} \Bigg\{ & \frac{1+(2\pi\alpha' B^E)^2}{\nu} \partial_1 X^i_a \partial_1 X^i_a + \nu \partial_2 X^i_a \partial_2 X^i_a - \\ & - 4\pi\alpha' B^E \partial_1 X^i_a \partial_2 X^i_a + g_s^2 \alpha' \nu \left(\partial_1 A_{2a} - \partial_2 A_{1a}\right)^2 - \\ & - i2\alpha' \Bigg(\psi^T_a \partial_2 \psi_a + \frac{1}{\nu} \psi^T_a \gamma \partial_1 \psi_a - \\ & \qquad - \frac{2\pi\alpha' B^E}{\nu} \psi^T_a \partial_1 \psi_a \Bigg) \Bigg\} \end{aligned} \qquad (6.21)$$

and with the boundary conditions (6.19). We should then sum over topologically distinct configurations, characterized by the permutations P and Q and by the integers m_μ, n_μ. We shall not discuss the validity of the assumptions leading to this starting point in the present paper (see discussions in [14, 15]).

Here, the eigenvalues of the matrices of the matrix model are functions which will turn out to live on multiple (un-branched) covers of the torus on which the field theory in (6.21) lives. The rest o this section is devoted to showing that the combinatorics of combining the eigenvalues to form these covers can be solved and that the resulting sum over all topological sectors produces a partition function which coincides with the one found for the superstring in the previous section.

It is easy to show that in the case that we are considering, the gauge field, after absorbing the gauge transformations in (6.19), decouples as $g_s \to 0$. This fact is particular to the genus 1 system that we are specializing to here. At higher genera it is claimed that the gauge field part of the partition function produces the correct factor of $(g_s)^{2g-2}$ which weights the contribution of higher genus covers [17].

The partition function that we must compute thus decomposes into topological sectors which are characterized by the permutations of the eigenvalues,

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} e^{-N\beta/\sqrt{2}R} \sum_{P,Q} Z(P,Q)\,, \qquad (6.22)$$

(where we have suppressed the dependence on the integers in the boundary conditions for gauge fields). Here, for each N we have divided by contribution by the volume of the Weyl group, $N!$, which reflects the fact that the eigenvalues are defined up to a global permutation. (From the M-theory point of view, this factor gives Boltzmann statistics to the D0-branes.) The possibility of introducing different weights in the sum over pairs of commuting permutations has been discussed in [22, 23]. Some relationships between these combinatorics and sigma models on symmetric orbifolds has been discussed in [24, 50, 51]. We shall show in what follows that the correct genus 1 partition function for the type-IIA string emerges from (6.22) only when all of the pairs (P,Q) have the same weight.

6.2.1 How permutations determine the world-sheet metric

The basic combinatorics which obtains the genus 1 (un-branched) covers of the torus from the sum over compatible permutations P and Q was discussed in detail in [21]. Here, we shall review the essential parts of the argument.

Consider a generic set of fields obeying the boundary conditions

$$\begin{aligned}
\lambda_a(\sigma_1+1,\sigma_2) &= \lambda_{P(a)}(\sigma_1,\sigma_2)\,,\\
\lambda_a(\sigma_1,\sigma_2+1) &= (-1)^f \lambda_{Q(a)}(\sigma_1,\sigma_2)\,,
\end{aligned} \tag{6.23}$$

where $f=0$ for a Bose field and $f=1$ for a fermion. A given permutation P can be decomposed into cycles — i.e. subgroups of the N elements which cycle into each other under repeated application of P.

Consider the subset of the fields which occur in the cycles of length k of P. We relabel them so that P acts on them as

$$\begin{aligned}
\lambda_{a^{k\alpha}_n}(\sigma_1+1,\sigma_2) &= \lambda_{a^{k\alpha}_{n+1}}(\sigma_1,\sigma_2)\,,\\
\lambda_{a^{k\alpha n}}(\sigma_1,\sigma_2+1) &= (-1)^f \lambda_{a^{k\pi(\alpha)}_{n+s}}(\sigma_1,\sigma_2)\,.
\end{aligned} \tag{6.24}$$

Here α is a label that runs over all of the different cycles of P which have length k. k runs over all lengths of cycles of a given permutation P.

For each such cycle, α, we fuse these fields together into a single function which lives on k copies of the torus and is periodic in the sense that

$$\lambda_{a^{k\alpha}}(\sigma_1+k,\sigma_2) = \lambda_{a^{k\alpha}}(\sigma_1,\sigma_2)\,. \tag{6.25}$$

Then we consider what a commuting permutation Q can do to these fields. Since Q must commute with P, there are only two possibilities for the action of Q. First, Q can permute cycles of P which have equal size — so it can act as a permutation of the set of all cycles of P with length k for example $Q:\alpha\to\pi(\alpha)$. Second, it can implement a cyclic permutation of the elements within a cycle of P which in the notation above is a translation of σ^1, $Q:\sigma^1\to\sigma^1+s(k,\alpha)$. Thus, Q has the action

$$\lambda_{a^{k\alpha}}(\sigma_1,\sigma_2+1) = (-1)^f \lambda_{a^{k\pi(\alpha)}}(\sigma_1+s(k,\alpha),\sigma_2)\,. \tag{6.26}$$

Then, we consider the decomposition of the action of Q into cycles. Such a cycle of Q permutes a subset (say of length r) of the r_k length k-cycles of P into each other. For this cycle of cycles, we again fuse together r copies of the fields $\lambda_{a^{k\alpha}}(\sigma_1,\sigma_2)$ so that the resulting field lives of r_k copies of the torus and is periodic

$$\begin{aligned}
\lambda_{a^{k,r}}(\sigma_1+k,\sigma_2) &= \lambda_{a^{k,r}}(\sigma_1,\sigma_2)\,,\\
\lambda_{a^{k,r}}(\sigma_1,\sigma_2+r) &= (-1)^{fr} \lambda_{a^{k,r}}(\sigma_1+s,\sigma_2)\,,
\end{aligned} \tag{6.27}$$

where $s=\sum_\alpha s(k,\alpha) \mod k$ is the accumulated shift for the r elements in the cycle of Q.

The space on which coordinates which are arguments of the field in (6.27) take values is a torus. The contribution to the path integral of this set of fields is denoted by

$$
\begin{aligned}
Z(r,k,s) = \int [dX^i][d\psi] \exp \Bigg\{ -\frac{1}{2}\int_0^r d\sigma_2 \int_{\frac{s}{r}\sigma_2}^{k+\frac{s}{r}\sigma_2} d\sigma_1 \times \\
\times \Bigg[\frac{\beta R(1+(2\pi\alpha' B^E)^2)}{\sqrt{2}(2\pi\alpha')^2}(\partial_1 X^i)^2 - B^E \partial_2 X^i \partial_1 X^i + \\
+ \frac{\sqrt{2}}{\beta R}(\partial_2 X^i)^2 - i\frac{\beta R}{\sqrt{2}2\pi^2\alpha'}\psi^T\gamma\cdot\partial_1\psi + \\
+ i2\alpha' B^E \psi^T \partial_1 \psi - \frac{i}{\pi}\psi^T\partial_2\psi \Bigg] \Bigg\},
\end{aligned}
\tag{6.28}
$$

where the integration region is a torus. The boundary conditions for the Bose fields are

$$
\begin{aligned}
X^i(\sigma_1 + k, \sigma_2) &= X^i(\sigma_1, \sigma_2), \\
X^i(\sigma_1, \sigma_2 + r) &= X^i(\sigma_1 + s, \sigma_2)
\end{aligned}
\tag{6.29}
$$

and for the Fermi fields are

$$
\begin{aligned}
\psi(\sigma_1 + k, \sigma_2) &= \psi(\sigma_1, \sigma_2), \\
\psi(\sigma_1, \sigma_2 + r) &= (-1)^r \psi(\sigma_1 + s, \sigma_2).
\end{aligned}
\tag{6.30}
$$

It is possible to change the coordinates in the action (6.28) so that the integration region is the square torus $(\sigma_1, \sigma_2) \in ([0,1),[0,1))$. The appropriate coordinate transformation is

$$
\begin{aligned}
\sigma_1' &= \frac{\sigma_1}{k} - \frac{s\sigma_2}{kr}, \\
\sigma_2' &= \frac{\sigma_2}{r}.
\end{aligned}
\tag{6.31}
$$

Then, the action becomes

$$
S = \frac{1}{4\pi\alpha'}\int_0^1 d\sigma_2 \int_0^1 d\sigma_1 \sqrt{g}\left(g^{\alpha\beta}\partial_\alpha \vec{X}\cdot\partial_\beta\vec{X} - i2\alpha'\bar{\psi}^i\gamma^a e_a^\alpha \partial_\alpha \psi^i \right), \tag{6.32}
$$

where

$$
g_{\alpha\beta} = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \tag{6.33}
$$

with

$$
\tau = \frac{s}{k} + \tau_0 \frac{r}{k}, \tag{6.34}
$$

where τ_0 is the parameter defined in (1.2) which was the Teichmüller parameter of the basic underlying torus of DLCQ strings.

e_a^α a zweibein which corresponds to the metric $g_{\alpha\beta}$. Now, the boundary conditions are

$$\begin{aligned}
X^i(\sigma_1+1,\sigma_2) &= X^i(\sigma_1,\sigma_2)\,, \\
X^i(\sigma_1,\sigma_2+1) &= X^i(\sigma_1,\sigma_2)\,, \\
\psi(\sigma_1+1,\sigma_2) &= \psi(\sigma_1,\sigma_2)\,, \\
\psi(\sigma_1,\sigma_2+1) &= (-1)^r\psi(\sigma_1,\sigma_2)\,.
\end{aligned} \tag{6.35}$$

Note that the boundary condition for the Fermi field still depends on r.

We have thus obtained the superstring partition function where the world-sheet is a fixed torus with Teichmüller parameter (6.34). This worldsheet is an un-branched cover of an underlying torus which has Teichmüller parameter τ_0 given in eq. (1.2) and coinciding with the basic parameter that we identified for the superstring.

It is now necessary to show that the sum over all compatible permutations P and Q produce the sum over genus 1 worldsheets which we found for the superstring. The relevant arguments are given in ref. [21] and we refer the reader there for the details. An essential piece of the derivation is to show the exponentiation of the sum over connected parts. Then, for a connected parts, the sum over the integers k, r, s in the domain which characterize the permutations produces the sum over discrete Teichmüller parameters which we found for the superstring at genus 1. In that sum, after replacing the euclidean B^E by the Minkowski field B, we obtain an expression for the free energy of the matrix string theory which precisely coincides with the superstring partition function which we found in (1.16).

7. Hagedorn temperature

7.1 Bosonic string

In bosonic string theory, the Hagedorn temperature is the temperature at which a certain winding modes become massless, indicating a phase transition. We will investigate the Hagedorn transition analyzing the mass shell condition, looking for the temperature at which a new tachyon, besides the one already present at zero temperature, appears [38].

Let us recall the expression for the world-sheet action in euclidean space in a B^E-field background.

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_\alpha X^\mu\partial^\alpha X_\mu - 2\pi i\alpha' B^E_{\mu\nu}\epsilon^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X^\nu\right). \tag{7.1}$$

Having compactified the time dimension to circumference β and the light-cone direction, one has

$$
\begin{aligned}
X^0 &= x^0 - (2\alpha')p^0 i\tau + \left(\frac{m\beta}{\pi} + i\sqrt{2}Rp\right)\sigma + (\text{oscillators})\,, \\
X^1 &= x^1 - (2\alpha')p^1 i\tau + \sqrt{2}Rp\sigma + (\text{oscillators})\,, \\
X_T^i &= x_T^i - 2\alpha' P_T^i i\tau + (\text{oscillators})\,.
\end{aligned} \tag{7.2}
$$

The momenta in the 0 and 1 directions are quantized according to

$$
\begin{aligned}
P^0 &\equiv -\int_0^\pi \frac{\partial \mathcal{L}}{\partial i\partial_\tau X^0} d\sigma \\
&= p^0 - B^E \pi\sqrt{2}Rp = \frac{2\pi n}{\beta}\,, \\
P^1 &\equiv -\int_0^\pi \frac{\partial \mathcal{L}}{\partial i\partial_\tau X^1} \\
&= p^1 + B^E\beta m + iB^E\sqrt{2}Rp = -\frac{\sqrt{2}k}{R} - i\frac{2\pi n}{\beta}\,.
\end{aligned} \tag{7.3}
$$

We then have

$$
\begin{aligned}
p^0 &= \frac{2\pi n}{\beta} + B^E\pi\sqrt{2}Rp\,, \\
p^1 &= -\frac{\sqrt{2}k}{R} - i\frac{2\pi n}{\beta} - B^E\beta m - iB^E\sqrt{2}Rp\,.
\end{aligned} \tag{7.4}
$$

The mass-shell condition becomes

$$
\begin{aligned}
M^2 \equiv -P_T^i P_T^i = \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) + \frac{m^2\beta^2}{4\pi^2\alpha'^2} + \frac{2k^2}{R^2} + \beta^2(B^E)^2 m^2 + \\
+ \frac{2\sqrt{2}B^E\beta}{R} km + 2\pi i\left[\sqrt{2}\beta Rmp\left(\frac{1}{4\pi^2\alpha'^2} + (B^E)^2\right) + \frac{2\sqrt{2}}{R\beta}kn\right] + \\
+ 4\pi i B^E\,(nm + kp)\,.
\end{aligned} \tag{7.5}
$$

The unfamiliar imaginary terms in this equation can be understood by considering that when M^2 is exponentiated to obtain the partition function, and summed over the integers, these terms will provide periodic delta functions that constraint the Teichmüller parameter. As a matter of fact there is an alternative way to obtain the mass spectrum. One can consider the torus path integral for the partition function. The sum over path integral topological sectors with the winding mode action given by (3.15) is related by Poisson re-summation to the partition function for the shifted spectrum. By first Poisson re-summing over n and then over q we get

$$\sum_{m,n,p,q} \exp\left[-\frac{\beta^2}{4\pi\alpha'\tau_2}(m\bar\tau - n)(\tau m - n) + 2\pi i \frac{\sqrt{2}\beta R}{4\pi\alpha'\tau_2}\times \right.$$
$$\left. \times\left(nq + |\tau|^2 mp - \tau_1(mq + np)\right) - 2\pi i\sqrt{2}\beta R\frac{B^E}{2}(mq - np)\right] =$$
$$= \frac{\sqrt{2}2\pi\alpha'\tau_2}{\beta R}\sum_{m,k,p,s}\exp\left[-\pi\tau_2\alpha'\left(\left(\frac{m\beta}{2\pi\alpha'}\right)^2 + \left(\frac{\sqrt{2}k}{R} + \beta B^E m\right)^2 + \right.\right.$$
$$+ 2\pi i\left(\sqrt{2}\beta R m p\left((B^E)^2 + \frac{1}{4\pi^2\alpha'^2}\right) - \right.$$
$$\left.\left. - \frac{2\sqrt{2}}{R\beta}sk + 2B^E(kp - ms)\right)\right) +$$
$$\left. + 2\pi i\tau_1(kp + ms)\right] \tag{7.6}$$

This formula clearly reproduces the topological part of the mass shell condition (7.5), when $s \to -n$, and of the constraint $L_0 - \tilde{L}_0 = 0$, which reads

$$L_0 - \tilde{L}_0 = nm - kp + N - \tilde{N} = 0 \tag{7.7}$$

with N and $\tilde{N}$ being the left- and right-moving mode number operators,

$$N = \sum_{n=1}^{\infty} : \alpha_n^\mu \cdot \alpha_{-n}^\mu :$$
$$\tilde{N} = \sum_{n=1}^{\infty} : \tilde\alpha_n^\mu \cdot \tilde\alpha_{-n}^\mu : \tag{7.8}$$

From (7.6) we then get the following form for the mass-shell condition

$$M^2 = \frac{2}{\alpha'}\left(N + \tilde{N} - 2\right) + \left(\frac{m\beta}{2\pi\alpha'}\right)^2 + \left(\frac{\sqrt{2}k}{R} + m\beta B^E\right)^2 \tag{7.9}$$

and, by summing over p and s, the periodic delta-functions

$$\delta\left[\pi\alpha'\tau_2\left(\sqrt{2}\beta R m\left((B^E)^2 + \frac{1}{4\pi^2\alpha'^2}\right) + 2B^E k\right) - \tau_1 k + r\right]$$
$$\delta\left[2\pi\alpha'\tau_2\left(\frac{\sqrt{2}}{R\beta}k + mB^E\right) + \tau_1 m - n\right], \tag{7.10}$$

where the sums over r and n have been omitted. Equation (7.10) provides the discretized Teichmüller parameter τ. As can be easily seen τ has again the form

$$\tau = \frac{n(1 - 2\pi i\alpha' B^E) - ir\nu}{m(1 - 2\pi i\alpha' B^E) - ik\nu}.$$

So that it coincides with the one defined in (3.32) when r is identified with s.

Looking for a state of $N = \tilde{N} = 0$, we must set the quantized momenta to zero, i.e. $n = k = 0$, and to find something "stringy", $m = \pm 1$. Consequently, in Minkowski space, M^2 becomes

$$M^2 = -\frac{4}{\alpha'} + \frac{\beta^2}{4\pi^2\alpha'^2} - \beta^2 B^2 \,. \tag{7.11}$$

We see that these modes become tachyonic when

$$\beta^2 = \frac{16\pi^2\alpha'}{1-(2\pi\alpha' B)^2} \tag{7.12}$$

which gives the Hagedorn temperature in the presence of a B-field.

For $B \to 0$ we recover the well-known result [38]

$$T_H = \frac{1}{4\pi\sqrt{\alpha'}} \,. \tag{7.13}$$

7.2 Superstring

In this section, we will extend the preceding discussion of the Hagedorn transition for bosonic closed string to the case of a supersymmetric type-II superstring in a constant B-field.

The description of supersymmetric string theories in the NSR formalism requires the sum over spin structures in addition to summing over windings. For type-II superstrings we must consider the contribution of both right-moving and left-moving NSR fermions. At genus one there are four spin structures for the right- and left-moving fermions. These spin structures are $(+,+)$, $(+,-)$, $(-,+)$, $(-,-)$, where the first sign refers to the σ_1 direction and the second to the σ_2 direction. Thinking of σ_1 and σ_2 as the world-sheet space and time direction, one has that space-time bosons (R sector) correspond to $(+,+)$ and $(+,-)$ and space-time fermions (NS sector) correspond to $(-,+)$ and $(-,-)$.

In [38], is explained that the effective string theory that governs the high-temperature limit of type-II superstrings is a theory in which one sums over all spin structures L, imposing that the right-moving spin structure is the same as the left-moving one. This is a ghost-free and modular invariant but tachyonic ten-dimensional string theory.

In order to study the Hagedorn temperature for type-II superstring, we must look for a state which is in the NS-NS sector, because only in this case the tachyon appears.

As shown in the superstring section, the B-dependent contribution comes from the bosonic part of the action. Thus the formula for the mass-shell condition is similar to that obtained for the bosonic string, with N and $\tilde{N}$ being respectively the right- and left-moving number operators of bosons and fermions, and with the appropriate value for the normal ordering constant (which is $1/2$ for both right and left sectors)

$$M^2 = \frac{2}{\alpha'}\left(N + \tilde{N} - 1\right) + \left(\frac{m\beta}{2\pi\alpha'}\right)^2 + \left(\frac{\sqrt{2}k}{R} + i\beta Bm\right)^2 . \qquad (7.14)$$

We are looking for a state with $N = \tilde{N} = 0$, so we must set the quantized momenta to zero, and, to find something "stringy", $m = \pm 1$. Except for the presence of the B-field in the mass-shell condition, all the discussion proceeds as in [38]. In Minkowski space the mass-shell condition becomes,

$$M^2 = -\frac{2}{\alpha'} + \frac{\beta^2}{4\pi^2\alpha'^2} - \beta^2 B^2 . \qquad (7.15)$$

These modes become tachyonic when

$$\beta^2 = \frac{8\pi^2\alpha'}{1 - (2\pi\alpha' B)^2} \qquad (7.16)$$

which gives the Hagedorn temperature in the presence of a B-field.

For $B \to 0$ we recover the well-known result [38]

$$T_H = \frac{1}{2\pi\sqrt{2\alpha'}} . \qquad (7.17)$$

It is possible to reproduce the result given in (7.16) by studying where the free energy in the Green-Schwarz formulation of the superstring diverges.

8. Discussion

What we have examined in this paper is an interesting coincidence between the set of all eigenvalues of simultaneously diagonalizable matrices which live on a torus and the set of Riemann surfaces that appear in the path integral for the string partition function. In both cases they are the sets of branched covers of a basic torus. We have found the further fact that the basic torus is the same in the case of matrix theory and the type-II superstring. We also find that the one loop, or genus one, contribution in both cases are identical in every detail.

It would be interesting to extend this correspondence to higher genera. We have done this partially by showing that the set of branched covers of the torus is what appears in the sum over Riemann surfaces in the string partition function. In this case, the higher genus covers can be obtained by gluing tori together along branch cuts. Analogous to this, it should be possible to show that a higher genus contribution to the string partition function can be obtained by taking correlators of the DVV vertex operators in the free matrix string model. This would potentially establish the appearance of string perturbation theory within the matrix model to all orders in the genus expansion.

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The effect of a B-field on the Hagedorn temperature that we have found is interesting. T_H depends on B and is lowered by increasing B, but it does not depend on the light-cone compactification radius R. This means that, for non-zero B, the DLCQ procedure has an infinite R limit which remembers the compactification and still feels the effect of B. This puzzling behavior needs further investigation. It leaves open the possibility that the DLCQ and at finite temperature is inconsistent in the sense that B doesn't decouple properly as the physically interesting limit $R \to \infty$ is taken.

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References

[1] G. Grignani, P. Orland, L.D. Paniak and G.W. Semenoff, *Matrix theory interpretation of dlcq string worldsheets*, *Phys. Rev. Lett.* **85** (2000) 3343 [hep-th/0004194].

[2] T. Banks, W. Fischler, S.H. Shenker and L. Susskind, *M-theory as a matrix model: a conjecture*, *Phys. Rev.* **D 55** (1997) 5112 [hep-th/9610043].

[3] L. Susskind, *Another conjecture about M(atrix) theory*, hep-th/9704080.

[4] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[5] N. Seiberg, L. Susskind and N. Toumbas, *Strings in background electric field, space/time noncommutativity and a new noncritical string theory*, *J. High Energy Phys.* **06** (2000) 021 [hep-th/0005040].

[6] I.R. Klebanov and J. Maldacena, *1+1 dimensional NCOS and its* $\mathrm{U}(N)$ *gauge theory dual*, hep-th/0006085.

[7] U.H. Danielsson, A. Guijosa and M. Kruczenski, *IIA/B, wound and wrapped*, *J. High Energy Phys.* **10** (2000) 020 [hep-th/0009182].

[8] J. Gomis and H. Ooguri, *Non-relativistic closed string theory*, hep-th/0009181.

[9] U.H. Danielsson, A. Guijosa and M. Kruczenski, *Newtonian gravitons and D-brane collective coordinates in wound string theory*, *J. High Energy Phys.* **03** (2001) 041 [hep-th/0012183].

[10] W. Taylor, *M(atrix) theory: matrix quantum mechanics as a fundamental theory*, hep-th/0101126.

[11] I.W. Taylor, *D-brane field theory on compact spaces*, *Phys. Lett.* **B 394** (1997) 283 [hep-th/9611042].

[12] L. Motl, *Proposals on nonperturbative superstring interactions*, hep-th/9701025.

[13] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Matrix string theory*, *Nucl. Phys.* **B 500** (1997) 43 [hep-th/9703030].

[14] T. Wynter, *Gauge fields and interactions in matrix string theory*, *Phys. Lett.* **B 415** (1997) 349 [hep-th/9709029].

[15] T. Wynter, *Anomalies and large-N limits in matrix string theory*, *Phys. Lett.* **B 439** (1998) 37 [hep-th/9806173].

[16] S.B. Giddings, F. Hacquebord and H. Verlinde, *High energy scattering and D-pair creation in matrix string theory*, *Nucl. Phys.* **B 537** (1999) 260 [hep-th/9804121].

[17] G. Bonelli, L. Bonora and F. Nesti, *String interactions from matrix string theory*, *Nucl. Phys.* **B 538** (1999) 100 [hep-th/9807232].

[18] G. Bonelli, L. Bonora, F. Nesti and A. Tomasiello, *Matrix string theory and its moduli space*, *Nucl. Phys.* **B 554** (1999) 103 [hep-th/9901093].

[19] T. Wynter, *High energy scattering amplitudes in matrix string theory*, *Nucl. Phys.* **B 580** (2000) 147 [hep-th/9905087].

[20] P. Brax, *The supermoduli space of matrix string theory*, *J. High Energy Phys.* **07** (2000) 020 [hep-th/9912103].

[21] G. Grignani and G.W. Semenoff, *Thermodynamic partition function of matrix superstrings*, *Nucl. Phys.* **B 561** (1999) 243 [hep-th/9903246].

[22] M. Billó, M. Caselle, A. D'Adda and P. Provero, *Matrix string states in pure 2d Yang-Mills theories*, *Nucl. Phys.* **B 543** (1999) 141 [hep-th/9809095].

[23] M. Billó, A. D'Adda and P. Provero, *Branched coverings and interacting matrix strings in two dimensions*, hep-th/0103242.

[24] H. Fuji and Y. Matsuo, *Open string on symmetric product*, *Int. J. Mod. Phys.* **A 16** (2001) 557 [hep-th/0005111].

[25] I.K. Kostov and P. Vanhove, *Matrix string partition functions*, *Phys. Lett.* **B 444** (1998) 196 [hep-th/9809130].

[26] A. Connes, M.R. Douglas and A. Schwarz, *Noncommutative geometry and matrix theory: compactification on tori*, *J. High Energy Phys.* **02** (1998) 003 [hep-th/9711162].

[27] M.R. Douglas and C. Hull, *D-branes and the noncommutative torus*, *J. High Energy Phys.* **02** (1998) 008 [hep-th/9711165].

[28] Y.-K.E. Cheung and M. Krogh, *Noncommutative geometry from 0-branes in a background B-field*, *Nucl. Phys.* **B 528** (1998) 185 [hep-th/9803031].

[29] N. Seiberg, *Why is the matrix model correct?*, *Phys. Rev. Lett.* **79** (1997) 3577 [hep-th/9710009].

[30] T.H. Buscher, *Path integral derivation of quantum duality in nonlinear sigma models*, *Phys. Lett.* **B 201** (1988) 466.

[31] T.H. Buscher, *A symmetry of the string background field equations*, *Phys. Lett.* **B 194** (1987) 59.

[32] A. Sen, *D0-branes on T_n and matrix theory*, *Adv. Theor. Math. Phys.* **2** (1998) 51 [hep-th/9709220].

[33] R. Schiappa, *Matrix strings in weakly curved background fields*, hep-th/0005145.

[34] I.W. Taylor and M.V. Raamsdonk, *Multiple Dp-branes in weak background fields*, *Nucl. Phys.* **B 573** (2000) 703 [hep-th/9910052].

[35] R. Hagedorn, *Statistical thermodynamics of strong interactions at high-energies*, *Nuovo Cim. Suppl.* **3** (1965) 147.

[36] B. Sathiapalan, *Vortices on the string world sheet and constraints on toral compactification*, *Phys. Rev.* **D 35** (1987) 3277.

[37] Y.I. Kogan, *Vortices on the world sheet and string's critical dynamics*, *Sov. Phys. JETP Lett.* **45** (1987) 709.

[38] J.J. Atick and E. Witten, *The hagedorn transition and the number of degrees of freedom of string theory*, *Nucl. Phys.* **B 310** (1988) 291.

[39] J. Ambjørn, Y.M. Makeenko and G.W. Semenoff, *Thermodynamics of D0-branes in matrix theory*, *Phys. Lett.* **B 445** (1999) 307 [hep-th/9810170].

[40] J. Ambjørn, Y.M. Makeenko, G.W. Semenoff and R.J. Szabo, *String theory in electromagnetic fields*, hep-th/0012092.

[41] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory*, Cambridge University Press, Cambridge U.K., 1987

[42] K.H. O'Brien and C.I. Tan, *Modular invariance of thermopartition function and global phase structure of heterotic string*, *Phys. Rev.* **D 36** (1987) 1184.

[43] B. McClain and B.D.B. Roth, *Modular invariance for interacting bosonic strings at finite temperature*, *Comm. Math. Phys.* **111** (1987) 539.

[44] J. Polchinski, *Evaluation of the one loop string path integral*, *Comm. Math. Phys.* **104** (1986) 37.

[45] I.W. Taylor and M.V. Raamsdonk, *Multiple D0-branes in weakly curved backgrounds*, *Nucl. Phys.* **B 558** (1999) 63 [hep-th/9904095].

[46] A. Bilal, *Dlcq of M-theory as the light-like limit*, *Phys. Lett.* **B 435** (1998) 312 [hep-th/9805070].

[47] A. Bilal, *A comment on compactification of M-theory on an (almost) light-like circle*, *Nucl. Phys.* **B 521** (1998) 202 [hep-th/9801047].

[48] S. Hyun and Y. Kiem, *Background geometry of DLCQ M-theory on a p-torus and holography*, *Phys. Rev.* **D 59** (1999) 026003 [hep-th/9805136].

[49] D.J. Gross, R.D. Pisarski and L.G. Yaffe, *QCD and instantons at finite temperature*, *Rev. Mod. Phys.* **53** (1981) 43.

[50] G. Arutyunov, S. Frolov and A. Polishchuk, *On Lorentz invariance and supersymmetry of four particle scattering amplitudes in $S^N R^8$ orbifold sigma model*, *Phys. Rev.* **D 60** (1999) 066003 [hep-th/9812119].

[51] G.E. Arutyunov and S.A. Frolov, *Four graviton scattering amplitude from $S^N R^8$ supersymmetric orbifold sigma model*, *Nucl. Phys.* **B 524** (1998) 159 [hep-th/9712061].